HIGH SCHOOL MATHEMATICS

FIRST COURSE

TEACHERS' EDITION

UNIT THREE 1957-58

EQUATIONS

UNIVERSITY OF ILLINOIS
COMMITTEE ON
SCHOOL MATHEMATICS

Unit 3

EQUATIONS

3.01 A report on a ticket sale.--Some problems are difficult to solve if you use only arithmetic.

3.02 The number line.--You can draw "pictures" of equations and inequalities on a number line.

3.03 Equations.--You can solve many equations by using your knowledge of arithmetic.

3.04 Formulas.--Each time you use a formula to solve a problem you must also solve an equation.

3.05 Solving more difficult equations.--There are two transformation principles which help you solve equations.

3.06 Using pronumerals to solve problems.--Algebraic expressions and equations make it easy to solve problems and puzzles.

3.07 Equations with different pronumerals.--You can derive other formulas from a given formula by using the transformation principles.

UICSM
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TEACHERS COMMENTARY

Introduction

This unit has sections which are delightful to teach, and sections which are among the most difficult to teach. Students learn how to solve equations, and then they are expected to use this skill in solving the familiar "verbal" problem.

Equation-solving is a skill which students enjoy acquiring, and in this unit, you will be agreeably surprised by the originality many students show in the introductory and informal work on this topic. Probably, the best way to acquire a high degree of skill in solving equations is to solve lots of equations. We have included long lists of exercises in the student's text, and supplementary lists in the Commentary. Most of these supplementary equations have integral roots. We want students to solve many equations, and they won't have time to do this if they must check too many fractional roots. You will note also that many of the supplementary equations require considerable work in expanding and simplifying algebraic expressions. This practice will help maintain the skills developed in Unit 2.

Apart from their need for proficiency in the mechanics of equation-solving, students need to understand what is meant by 'solving an equation'. To solve an equation is not to write a sequence of steps ending with '$x = \ldots$'. When one solves an equation, he seeks all those numbers which satisfy the equation. The methods which a student uses to do this are many. In the early part of the unit, he is expected to invent his own methods for solving equations [and inequations]. This work is informal and must be kept informal in order to drive home the point that one is seeking roots when he solves an equation. In the formal work on transforming equations [section 3.05], we still keep uppermost the notion of seeking roots but now we make use of the concept of pairs of equivalent equations. You must spend time in class making sure that the student understands that deriving equivalent equations is a useful method because the roots of the given equation can be obtained by finding the roots of a derived equation.

After students have shown a reasonable understanding of this notion, they should be drilled sporadically until they can find the roots of an equation rapidly and mechanically. Every so often they should encounter in their drill work an equation which has no roots, or an equation which has each real number as a root. And every so often, a student should be required to give a step-by-step justification of the procedure he is using in solving an equation.

The troublesome part of this unit is the section on verbal problems. All mathematics teachers can remember that in their student days the work on verbal problems was fun. Yet, it seems to be difficult to get most of our students to the place where they can solve verbal problems with the same facility that they demonstrate in solving equations. Part of the difficulty resides in the fact that the student and the teacher are frequently working at cross-purposes. The student is eager to find an answer to the problem [assuming that he wants to solve the problem at all], and he is not overly concerned with the method used in finding the answer. The teacher, on the other hand, is anxious about the student's failing to use an equation in obtaining his answer. We submit that the student is very much like the applied mathematician in his approach to problem-solving. "Use whatever method you can to find the answer." If you are faced with solving many problems of the same kind, devise a general procedure to find the answer. The contribution that Unit 3 makes to the student is to add another weapon to his arsenal of problem-solving methods. Indeed, it is a very powerful weapon but it is not the only weapon. We must show students the power of this weapon by using it to solve really tough problems early in the game, and then telling him that he can gain skill in its use by practicing on problems which may be simple enough to yield to methods which he learned in earlier grades. Admittedly, this is an artificial situation, and it may be helpful to let students know that you regard it as artificial. He won't object provided that the pay-off is not delayed unnecessarily. [This is one of the reasons why we have placed traditionally difficult verbal problems rather early in the list of exercises (pages 3-55 ff).]

* * *

It is in this unit that the student, for the first time in the UICSM program, encounters the notion of set. In the appropriate places in the Commentary we ask you to augment the student materials by formalizing the language just a bit, principally by the introduction of the set-abstraction operator notation. The underlying theme in bringing in the notation is that it provides abbreviations for long-winded expressions. In Unit 4 we ask you to continue this augmentation. In the next revision we plan to add these things to the student materials, for we think that the student can handle the additional symbols, and that with this background he will be better prepared for the use of set-concepts in SECOND COURSE.

Some of your students will be able to solve this problem by "arithmetic". Check their solutions individually in class without revealing them to the rest of the students. Our purpose here is to give the students a moderately difficult problem so that they will discover that considerable ingenuity is required to solve the problem by methods learned in earlier grades. It is likely that most of your students will be unable to solve the problem. Do not solve it for them. Instead, promise them that by the time they finish this unit, they will be able to solve problems like this without much effort at all. It is best to do this work and the exercises on page 3-2 in class so that students will not be unnecessarily tempted to get help from home. If a student discovers an algebraic method, acknowledge his success but do not try to explain the method to the class.

* * *

You may be surprised by the cleverness of the solutions offered by some of your students. Here is a typical arithmetic solution.

> An adult ticket costs more than a student ticket. If 167 student tickets were sold, the total collected would be (167×39) cents or \$65.13. But, a total of \$79.17 was collected. So the difference between the totals, \$14.04, must be due to the fact that some adult tickets were sold. An adult ticket costs 27 cents more. Therefore, divide 14.04 by .27. You get 52. This is the number of adult tickets that were sold.

This solution is just for your reference; do not present it to your class.

3.01 <u>A report on a ticket sale.</u>--Betty Morris, who is chairman of the committee in charge of selling tickets for the play at Zabranchburg High School, gave a financial report to the Student Council:

> "The ticket committee sold 167 tickets to the school play. We charged \$.66 for an adult ticket and \$.39 for a student ticket and we collected a total of \$79.17."

John Sanders, a member of the Student Council, had been critical of the expensive advertising used to interest adults in coming to the play. After Betty gave her report, he said,

> "I wonder if all of our advertising was worth the money. I didn't see many adults at the play. Just how many adults did attend the play, Betty?"

The Council president asked Betty if she could tell them the number of adult tickets sold. Betty said that she couldn't give this information immediately because her records were at home. If the Council didn't mind waiting, she would call home and ask her mother to read the figures to her over the telephone. Bill Smith, another Council member, said,

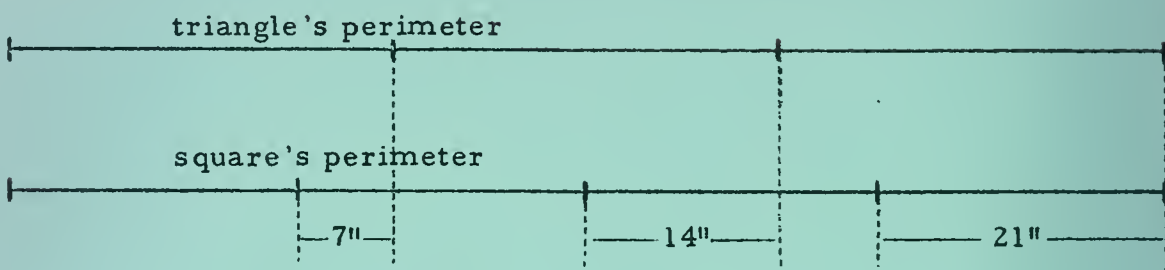
> "That won't be necessary, Betty. As soon as John asked his question, I used the information you gave us and computed the number of adult tickets sold. It was easy to find the answer."

<u>Can you tell how many adult tickets were sold?</u>

The problem Bill solved is not an easy one if you try to solve it by methods you have used in earlier grades. It can be solved by those methods, but there is a faster method which makes this problem a very easy one. You will learn this faster method and be able to solve even much more complicated problems in this unit. Before you learn this method, you need to have more practice with algebraic expressions.

Typical arithmetic solutions.

Exercise 1.



One side of the triangle is 7 inches longer than one side of the square. The total of two sides of the triangle is 14 inches longer than the total of two sides of the square. The total of three sides of the triangle is 21 inches longer than the total of three sides of the square. But, the length in inches of the fourth side of the square is equal to the difference between the perimeter of the triangle and the length in inches of the total of three sides of the square. Therefore, the length of a side of the square is 21 inches.

Exercise 2.

If, after multiplying the certain number by 3, I have to subtract 9 to get what I obtain by adding 4 to the certain number, then the certain number must be 13 less than 3 times itself. But, if subtracting 13 from three times a number gives you the number itself, then 13 must be equal to twice the number. So, the number is $6\frac{1}{2}$.

Exercise 3.

Tom was always 3 years older than Ed. Nine years ago Tom was 3 years older and twice as old. So, 9 years ago Ed was 3 years old and Tom was 6 years old. Tom is 15 and Ed is 12.

EXERCISES

Try to solve the following problems in any way you can.

1. A square and an equilateral triangle have equal perimeters. A side of the triangle is seven inches longer than a side of the square. What is the length of a side of the square?

2. I'm thinking of a certain number. If I add 4 to that number, I get a sum which is equal to what I get after I multiply that certain number by 3 and subtract 9. What number was I thinking of?

3. Tom is 3 years older than Ed. Nine years ago Tom was twice as old as Ed. How old is each boy now?

3.02 *The number line*.--At the end of Unit 1 you learned that directed numbers could be used as *coordinates* of points on a straight line, and that the points on a straight line could be considered as *graphs* of directed numbers. A straight line whose points are associated in that way with directed numbers is called a *number line*.

-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
T	G	A	F	C	P	Q	R	L	H	Z	B	E	K	J	M	D

EXERCISES

A. In each of the following exercises tell the location of the graph of the given number. (Use the diagram above.)

4. Use the simplest expressions you can to complete the following table.

	c(A)	c(B)	c(the mid-point of segment AB)
(a)	2	________	5
(b)	-6	________	-10
(c)	________	1	-3
(d)	2x	________	7x
(e)	________	3t	8t + 5
(f)	a + 3b	________	7a - 2b
(g)	2x - 3y + 5	________	-7x + 8y - 6
(h)	2 - 3(x - y)	________	5 - 7(3x - 4y)

* * *

[Supplementary Exercise 3(d) differs from those that precede it in that pronumeral expressions are used instead of numerals. Point out to the student that this exercise really requires him to complete the following sentence:

For every x, if c(A) = 3x and c(B) = 9x then
c(the mid-point of segment AB) = ________.]

* * *

The notation 'c(...)' is used in SECOND COURSE. It is pronounced as 'see of ...' .

Parts A and B [pages 3-2 through 3-4] should be worked and checked in class to be sure that students are proceeding correctly. As a home assignment you may want to give them these Supplementary Exercises.

1. If a point A on the number line has coordinate 7 and a point B has coordinate -3 then the mid-point of the segment AB has coordinate ________ .

2. Suppose the symbol 'c(P)' to be an abbreviation for the long expression 'the coordinate of the point P on the number line'. Now complete the following sentence:

   If c(A) = 7 and c(B) = -3 then c(the mid-point of segment AB) = ________ .

3. Use the simplest expressions you can to complete the following table.

	c(A)	c(B)	c(the mid-point of segment AB)
(a)	+9	+31	________
(b)	-2	-16	________
(c)	0	5	________
(d)	3x	9x	________
(e)	2k + 7	6k - 7	________
(f)	3a + 2b	7a - 16b	[Answer: 5a - 7b]
(g)	2(5 - x)	7(2 + 4x)	________
(h)	3 - 5r + 7t	11 - 7r - 21t	________
(i)	5 - (7 + 3n)	9n - (6 - 8n)	________
(j)	-(3 - x + 2y)	-(5 + 7x - 12y)	________

(continued on T. C. 3B)

Sample 1. $+6$

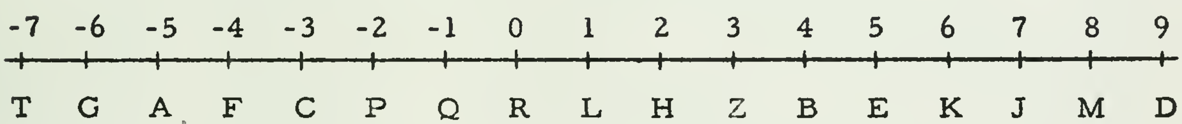
Solution. The graph of $+6$ is the point K.

Sample 2. $-3\frac{1}{3}$

Solution. The graph of $-3\frac{1}{3}$ is the point which is $\frac{1}{3}$ of the way from C to F.

1.	3	2.	$+7$	3.	-5
4.	0	5.	2.5	6.	$-5\frac{1}{4}$
7.	$+4\frac{9}{10}$	8.	-4.8	9.	4.99
10.	-0.1	11.	-1%	12.	$.1\%$
13.	600%	14.	$4 + (-3)$	15.	$\frac{1}{2} + \frac{1}{3}$
16.	6% of -150	17.	$3 \times 1\frac{1}{2}$	18.	$(-2) \div (-\frac{1}{2})$

B. Tell the coordinate of the given point in each of the following exercises. (Use the diagram below.)



Sample 1. The point A.

Solution. The coordinate of point A is -5.

Sample 2. The midpoint of the line segment LP.

Solution. The line segment is 3 units long. So, the midpoint is $1\frac{1}{2}$ units from L to P. Thus, the coordinate of the midpoint is $-\frac{1}{2}$.

Note that Exercise 15 is not answered correctly if the student gives the coordinate of only one point. He must give the coordinates of two points: -3 and $-\frac{1}{3}$. Needless to say, he should find these coordinates intuitively. Similarly, there are two points to be considered in Exercise 16. [Challenge the students to give a convincing argument that there are no more than two points which satisfy the description in Exercise 15.] Ask students why we shifted from the definite article 'The' to the indefinite article 'A'.

* * *

When the student sees a sentence such as '$x > 3$', he should not think of this sentence as a true statement or as a false statement. Instead, he should view it as something which requires completion. That is, the pronumeral 'x' should be regarded as a "receptacle" which awaits a numeral. It is only after the pronumeral has been replaced by a numeral that a student can say 'true' or 'false' to the resulting statement. Whenever a student becomes confused on this point, return to the use of frames.

* * *

You will note that the word 'expression' is used here as a catch-all term, as it is in certain other places in the text, and as has been pointed out in the Commentary for Unit 2 [T. C. 65B ff]. Since later in the unit we shall refer to the members of an equation as algebraic expressions, it is best <u>now</u> to refer to equations by 'equations' or by 'sentences' rather than by 'expressions'. Similarly, it is best to refer to inequalities by 'inequalities', or by 'inequations', or by 'sentences'.

-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
T	G	A	F	C	P	Q	R	L	H	Z	B	E	K	J	M	D

1. The point D.
2. The point T.
3. The point R.
4. The point C.
5. The point 1 unit to the right of A.
6. The point 2 units to the left of M.
7. The point halfway between D and F.
8. The mid-point of the line segment DT.
9. The mid-point of the line segment FH.
10. The point 40% of the way from A to L.
11. The point one third of the way from M to J.
12. The point one fourth of the way from P to A.
13. The point which is as far from C as it is from H.
14. The point between G and Z which is twice as far from G as it is from Z.
15. A point which is twice as far from L as it is from Q.
16. A point which is three times as far from P as it is from H.

SETS OF POINTS

Consider the expression:

$$x > 3$$

If we replace the pronumeral 'x' by a numeral, say, '5', the resulting statement:

$$5 > 3$$

and still talks about the set of dishes on display in the china closet even though there is only one member left.]

* * *

Example 2 can be troublesome. The dramatic approach may help. Here sits the sad sentence:

$$k > k + 1$$

waiting to meet each of all the numbers in the world, and ready to assign it to one of the two sets, the solution set of '$k > k + 1$' or the complement of the solution set. When all the numbers have introduced themselves and have been assigned to the proper set, we find that the solution set of '$k > k + 1$' has no members at all. It's just plain empty. But, we still talk about the solution set of '$k > k + 1$'. In fact, we call it the empty set. Why do we use the definite article 'the'? Because there is only one set like this. For example, the solution set of '$x + 5 = x$' is the same set as the solution set of '$k > k + 1$'. Each of these sentences has the empty set as its solution set. Why should anyone ever want to talk about such a thing as the empty set? Because it is convenient to be able to say that each sentence has a solution set. Of course, a picture of the locus of '$k > k + 1$', that is, a picture of the empty set is very easy to make:

locus of '$k > k + 1$'

No shading, no heavy dots.

The arrow indicates that all the points to the right of the point with coordinate 12 are in the picture of the locus. The "hollow" dot indicates that the point with coordinate 12 is not in the picture of the locus. Point out that the unshaded portion of the diagram is a picture of the complement of the locus of 'x + 3 > 15'. [Note that the text states that a locus is a set of points. If one regards the points of the number line as numbers (as we do in SECOND COURSE), it is permissible to let 'locus' have the apparent ambiguity.] Examples 1 and 2 should be handled in mildly dramatic fashion. In Example 1 the sentence to be satisfied is:

$$t = -2.$$

Again, all the numbers in the world introduce themselves to 't = -2', and each number is put into one of two sets. The solution set of 't = -2' consists of just one lonely number, -2. The complement of the solution set consists of all the other numbers. So, here we have a pair of sets with one of them consisting of a single member (work 'member' into the discussion). Sets with single members are quite common in mathematics. In fact, there is a special name for such sets. They are called singletons. A picture of the locus of 't = -2' is a single dot:

[One of Mrs. Catlow's students helped his fellows understand the difference between a singleton set and the member of this set by telling the story of how proud his mother is of a certain set of dishes she inherited. She had these dishes for many years, and, in the normal course of events, one after another of the dishes broke. Now there is just one cup left. Barney's mother is still proud of this set of dishes

(continued on T. C. 5C)

The words 'satisfy', 'set', and 'locus' are used repeatedly throughout Units 3 and 4, and in SECOND COURSE. It is worthwhile spending time on these words right now. You can "ham it up" a bit by telling the story of the sad sentence:

$$x + 3 > 15$$

who is waiting for numbers to satisfy it. Each number introduces itself to the sad sentence and asks, "Do I satisfy you?" When 7 asks the question, the sentence answers, "No." Why? Because, the new sentence '$7 + 3 > 15$' is false. But $27\frac{1}{2}$ gets the yes-answer because '$27\frac{1}{2} + 3 > 15$' is true. When all the numbers in the world (including the directed numbers) have been answered, we find that the sentence has separated these numbers into two sets, the set of those numbers which satisfy it and the set of those numbers which do not satisfy it. The students will tell you that in the "satisfying set" [technically, this is called the solution set of '$x + 3 > 15$', and you should introduce this terminology after using 'satisfying set' a few times] each number is larger than 12, and that in the other set [technically, it is called the complement of the solution set of '$x + 3 > 15$'] each number is either 12 or smaller than 12.

We can make a "picture" of the solution set [and, at the same time, of its complement] by using the number line diagram. When we want to make such a picture, we usually talk about making a picture of the locus of the sentence rather than of the solution set of the sentence. ['locus' comes from the Latin for 'place'; its plural is 'loci'.] Since each number in the locus of '$x + 3 > 15$' is the coordinate of a point on the number line diagram, if you darken each point which corresponds to a number in the locus, you get this picture:

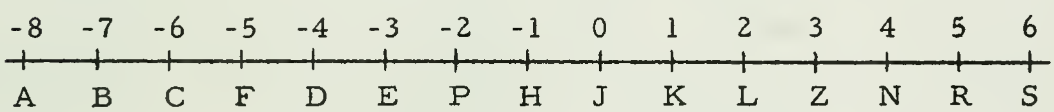
(continued on T. C. 5B)

is true. When the 'x' in 'x > 3' is replaced by a numeral for a number such that the resulting statement is true, the number is said to satisfy the expression 'x > 3'. There are many numbers which satisfy 'x > 3'. A few such numbers are $+3\frac{1}{2}$, +15, 8.7, and 900%. (There are also many numbers which do not satisfy 'x > 3'. What are some of these numbers?) Now, consider the set of all numbers which satisfy 'x > 3'. Suppose each of these numbers is regarded as the coordinate of a point on the number line. Then, there is a set of points on the number line each of whose coordinates satisfies 'x > 3'. This set of points is called the locus of 'x > 3'. For the number line shown below, the locus of 'x > 3' is the set of all points which are to the right of the point Z.

Note: The diagram shows only a piece of the number line. You should consider the line in the diagram as extending indefinitely in both directions. Thus, when we say that the locus of 'x > 3' is the set of points on the right of point Z, we mean to include, for example, the point whose coordinate is +1,000,000.

Example 1. Describe the locus of 't = -2'.

Solution. There is only one number which satisfies 't = -2'. This number is -2. Therefore, the set of points each of whose coordinates satisfies 't = -2' contains only one point. So, we say that the locus of 't = -2' has the point P as its only member.



Example 2. Describe the locus of 'k > k + 1'.

Solution. If we replace each 'k' in 'k > k + 1' by a numeral for the same number, we find that there is no number which satisfies 'k > k + 1'. Therefore, the set of points each of whose coordinates satisfies 'k > k + 1' does not contain any points. The set which has no members is called the empty set. So, we describe the locus of 'k > k + 1' by saying that the locus is the empty set.

7. $\overrightarrow{JD}$ 8. $\overset{\circ}{\overrightarrow{JD}}$ 9. $\overset{\circ}{\overrightarrow{DJ}}$

10. the complement of $\overrightarrow{KS}$. Of $\overrightarrow{PC}$. Of $\overset{\circ}{\overrightarrow{DK}}$.

V. 1. Write two sentences such that the solution set of one of them is the complement of the solution set of the other.

2. Write a sentence whose solution set is the complement of

(a) the solution set of '$x + 5 = 9$'

(b) the solution set of '$x \geq 17$'

(c) the solution set of '$k - 5 \neq 15$'

(d) the solution set of '$p < 12$'

(e) the solution set of '$x - 3 = x - 4$'

(f) the complement of the solution set of '$xx + 83x - 571 = 0$'

* * *

Supplementary exercises 4 and 5 of part III above are very important. In Exercise 4 we are looking for a sentence which is satisfied only by the number 0. This is quite different from looking, as in Exercise 5, for a sentence which is not satisfied by any number at all.

You will want students to demonstrate their understanding of the ideas you introduced in doing the following

Supplementary Exercises.

I. Which of the following sentences is satisfied by the number 8?

1. $3 + x = 11$
2. $5 + 3x = 64$
3. $2 - x = x + 3$
4. $9 - 2y > -5$
5. $3(a + 7) - 5(6 - a) = 7a - 1$
6. k is an even number
7. $3 + 2t < 5t - 9$
8. $x + 3 = 3 + x$
9. $y \div 0 = 8$

II. {5, -5} is the solution set of which of the following sentences?

1. $xx = 25$
2. $yy - 25 = 0$
3. $25 + kk = 0$
4. $m + 5 = 0$
5. $(t - 5)(t + 5) = 0$
6. x is the opposite of $-x$
7. $4 \square \square = 100$
8. $5 \triangle = -5$
9. $x + 3 \neq 9$
10. $5 + y \neq 10$

III. Write two sentences each of which has as solution set

1. { 6 }
2. { 3 }
3. { 3, - 3 }
4. { 3 - 3 }
5. the empty set
6. the set of all numbers

IV. Refer to the picture of the number line on page 3-6 and write a sentence whose locus is

1. { Z }
2. { J }
3. {C, S}
4. $\overrightarrow{ZS}$
5. $\overrightarrow{KA}$
6. $\overrightarrow{PH}$

(continued on T. C. 6D)

'{E}' is read as 'the set consisting of the point E'.] The locus of '$y^2 = +4$' can be described by: {P, L}.

The loci for Exercises 10, 11 and 12 cannot be described in precisely this fashion because they contain more than "just a few" members. However, there are convenient symbols which can be used instead of the cumbersome expressions of the form 'the set consisting of . . .'. These are symbols which we use in SECOND COURSE and which can be introduced here with profit. The appropriate symbol for Exercises 10, 11, and 12 is a symbol for a <u>half</u>-<u>line</u>. ["Define" 'half-line' as a name for each of the two things you get when you knock a point out of a straight line.] Thus, acceptable answers to Exercise 10 are:

$$\overrightarrow{DA},\quad \overrightarrow{DB},\quad \overrightarrow{DC},\ \text{and:}\ \overrightarrow{DF}.$$

The letter on the left corresponds to the point you knocked out of the line to get the two half-lines; the letter on the right corresponds to any one of the points in the appropriate one of the two half-lines. [The symbol '$\overrightarrow{DA}$' is read as 'the half-line from dee through ay', and could be read with the diagram in view as 'the set of points to the left of D'.] Now ask for short descriptions for Exercises 11 and 12.

To deepen understanding of the notion of half-line and to promote the careful use of the symbol, consider, with the students, a description of the locus of '$x \le -4$'. Ask them to suggest a symbol. Work up to: $\overset{\circ\!\!\longrightarrow}{DA}$, or any of: $\overset{\circ\!\!\longrightarrow}{DB}$, $\overset{\circ\!\!\longrightarrow}{DC}$, and: $\overset{\circ\!\!\longrightarrow}{DF}$. This set of points is called <u>a</u> <u>ray</u>. Elicit from students a correct reading of: $\overset{\circ\!\!\longrightarrow}{DA}$, and the significance of the difference between '$\circ\!\!\longrightarrow$' and '$\longrightarrow$'.

* * *

The first twelve exercises in Part A should first be worked orally. Note that the instructions ask for descriptions of loci. Since a locus is a set of points (or: numbers), a description of a locus should refer to a set. So, acceptable answers to Exercise 1 are:

the set consisting of the point with coordinate -3,

and :

the set consisting of the point E.

[The second of these answers is preferable here since in Part A we are trying to establish familiarity with the idea of connecting a picture with a sentence.]

[Of course, a labelled diagram is also acceptable as a description even though the reference to a set is not apparent.] Acceptable answers to Exercise 10 are:

the set of all points with coordinates less than -4,

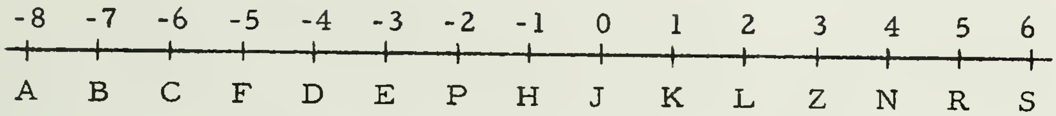
and:

the set of all points to the left of D.

When you have completed the oral discussion of these exercises, ask students to write the complete answers in their texts. Of course, there is not enough space to do easily a complete job of writing these answers. After students have griped their way through Exercise 3, ask, "How many are tired of writing 'The set consisting of so-and-so' "? You should get an enthusiastic response. This is the time to introduce a convenient notation. Announce to students that when mathematicians want to describe in writing a set which consists of just a few numbers, they do so by writing a pair of braces [say 'curly braces' if you want the first association they make to be correct], and writing between these braces names for the members of the set, the names being separated by commas. For example, a short and acceptable answer to Exercise 1 is: $\{E\}$. [The symbol

EXERCISES

A. Describe the locus of each of the following. (Use the diagram below.)



1. $x = -3$
2. $s = 0$
3. $z = -4$
4. $a = 33\frac{1}{3}\%$
5. $b = -6.5$
6. $-m = 2$
7. $\square = -7$
8. $-\square = -2.5$
9. $\triangle = -2$
10. $x < -4$
11. $+2 > y$
12. $k > -2\frac{1}{2}$

Sample. $1 < x < 5$

Solution. The expression '$1 < x < 5$' is an abbreviation for the expression:

$$1 < x \quad \text{and} \quad x < 5$$

A number which satisfies '$1 < x < 5$' is a number which satisfies both '$1 < x$' and '$x < 5$'. Therefore, the locus of '$1 < x < 5$' is the set of points between the points K and R. (Does the point K belong to the locus? Does the point R belong to the locus?)

(continued on next page)

Be sure that Exercise 49 is done in class, for you may need to review the fact that the directed numbers may be put into three separate categories: positive numbers, negative numbers, and $\{0\}$. The directed number 0 is neither positive nor negative.

* * *

You may want to give supplementary exercises to practice using these new symbols. Such exercises might take the form of those given on page 3-7 with the instructions that the student is merely to write names of geometric figures in describing the loci of the given sentences.

* * *

Note that we do not expect students to memorize the symbols which we suggested that you introduce. The purposes of this introduction are to prepare the student in a small way for some of the symbolism he will encounter in SECOND COURSE and to give him an opportunity to experience the labor-saving aspect of symbolism. If a student invents his own set of symbols for these geometric figures, please send a record of these symbols to us.

This is not bad, but should not be adopted since '$\overrightarrow{ZS}$ and $\overrightarrow{EA}$' is not conventionally used as a name of a set. [A proper symbol is: $\overrightarrow{ZS} \cup \overrightarrow{EA}$. But, there is little need to introduce ' $\cup$ ' at this point since ' $\cap$ ' cannot be brought in conveniently (and you would want to consider them together). Also, there is a briefer symbol available.] Ask students to describe the complement of the locus of 'xx > 9'. It is $\overline{EZ}$. Now, what is the complement of $\overline{EZ}$? It is the locus in question. We have a special symbol for the complement of a set. It is made by writing a tilde over a name of the set. Thus, the complement of $\overline{EZ}$ is named by:

$$\widetilde{\overline{EZ}}\,.$$

So, a brief answer to Exercise 29 would be written as above, and read as 'the complement of the segment ee zee'.

[Answers to Exercises 30 through 35 are: $\widetilde{\overline{EZ}}$, $\emptyset$, $\{F, R\}$, $\overline{FR}$, $\widetilde{\overline{FR}}$, and: $\emptyset$, respectively.]

A diagram for Exercise 36 consists of complete shading and an arrow at each end. An answer in words is 'the set of all points'. [A burst of applause for the student who says: the complement of the empty set, or who writes: $\widetilde{\emptyset}$.] A name for this figure is 'the line through A and S', and a brief symbol is '$\overleftrightarrow{AS}$'.

[Answers to Exercises 37 through 56 are: $\{R, H\}$, $\{K, R\}$, $\overleftrightarrow{AS}$, $\{K\}$, $\{\widetilde{J}\}$, $\emptyset$, $\{J\}$, $\emptyset$, $\overleftrightarrow{AS}$, $\overleftrightarrow{AS}$, $\overleftrightarrow{AS}$, $\{J\}$, $\overrightarrow{JS}$, $\overrightarrow{JA}$, $\overrightarrow{JA}$, $\overrightarrow{JS}$, $\{\widetilde{J}\}$, $\{J\}$, $\{\widetilde{J}\}$, and: $\overrightarrow{JA}$, respectively.]

(continued on T. C. 7D)

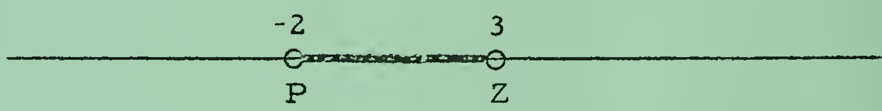
a straight line. For example, the figure in Exercise 13 is called *an interval*, and is symbolized by '$\overline{PZ}$' [or: '$\overline{ZP}$']. Now jump to Exercise 25, and ask for the answer "in words". A concise answer is: the set consisting of the point E, the point Z, and all the points between E and Z. Quite cumbersome. The students probably know that this geometric figure is called *a segment*. [They may have suggested this term for Exercise 13, at which time you probably contrasted Exercise 13 with Exercise 25 and gave the correct terms.] The figure in Exercise 25 is named by '$\overset{\circ\;\;\circ}{\overline{EZ}}$' [or. '$\overset{\circ\;\;\circ}{\overline{ZE}}$']. It should be easy to get students to tell the significance of the difference between '$\overline{\quad}$' and '$\overset{\circ\!-\!\!-\!\circ}{\quad}$'. Then return to Exercise 14 and proceed with oral reporting of answers. [Answers to Exercises 14 through 22 are: $\overline{FP}$, $\overline{JN}$, $\overline{CR}$, $\overline{HK}$, $\emptyset$, $\overrightarrow{NS}$, $\overrightarrow{FS}$, $\emptyset$, and: $\overrightarrow{ES}$, respectively.]

Exercises 23 and 24 offer new problems. Students may suggest 'segerval' and 'interment' as names for geometric figures of this type. [They will have no trouble inventing abbreviated names: '$\overset{\circ\;\;\;}{\overline{DR}}$' for Exercise 23 and '$\overset{\;\;\;\circ}{\overline{DR}}$' for Exercise 24.] Commonly used names are 'half-open interval', 'half-closed interval', 'half-open segment', and 'half-closed segment'.

The next exercise of interest is Exercise 29. The locus is hard to describe in words. You might get: the set consisting of those points which are to the right of Z and those points which are to the left of E. [Clearly, 'the set consisting of those points to the right of Z and to the left of E' is wrong since there are no points which fit this description.] Another answer you might get is: $\overrightarrow{ZS}$ and $\overrightarrow{EA}$.

(continued on T. C. 7C)

The exercises on this page afford many opportunities to emphasize the role of symbols as convenient abbreviations [and to foreshadow some of the work in SECOND COURSE]. All of these exercises should be assigned with the instructions to the student that he describe the locus by making a picture. You should work Exercises 13, 18, 23, and 25 as samples to establish conventions concerning drawings. It is unnecessary and deadening for students to make a copy of the diagram on page 3-7 for each exercise. [Perhaps you can get two or three students to volunteer to make on ditto masters 44 copies of this diagram, so that each student in the class can be provided with this "graph paper".] Certainly, an acceptable drawing for Exercise 13 is:



An acceptable answer for Exercise 18 is simply the phrase 'the empty set' or, better still, introduce the symbol: $\emptyset$, and let them use this as an answer.

An effective way to stimulate a need for convenient abbreviations is to ask students to read their "answers" aloud in class the day after they made the drawings. [Perhaps you assigned Exercises 13 through 32.] A student might say in answer to Exercise 13, "The set of all points between P and Z." This is relatively easy to say but is somewhat tedious to write. So, just as there are special names (and symbols) for the geometric figures, the half-line and the ray, there are special names for other geometric figures, sets of points which are parts of

(continued on T. C. 7B)

-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
A	B	C	F	D	E	P	H	J	K	L	Z	N	R	S

13. $-2 < x < 3$
14. $x > -5$ and $x < -2$
15. $0 < x < 4$
16. $-6 < n < +5$
17. $1 > t > -1$
18. $1 > s > 5$
19. $-3 < x > 4$
20. $-5 < x > -5$
21. $6 < x < 6$
22. $-x < 3$
23. $-4 \leq x < 5$
24. $-4 < x \leq 5$
25. $-3 \leq x \leq 3$
26. $6 \leq x \leq 6$
27. $xx = 9$
28. $xx < 9$
29. $xx > 9$
30. $xx \geq 9$
31. $kk = -4$
32. $k(-k) = -25$
33. $|x| < 5$
34. $|x| > 5$
35. $|x| < -5$
36. $|x| > -5$
37. $|k - 2| = 3$
38. $|3 - m| = 2$
39. $x + 1 = 1 + x$
40. $x - 1 = 1 - x$
41. $x \div x = 1$
42. $t \div t = 2$
43. $t = 2t$
44. $x - 2 = x - 3$
45. $\square + 2\,\square = 3\,\square$
46. $5g - 3g = 2g$
47. $3t \times 5t = 15tt$
48. $6 \times 3y = 9y$
49. $\square$ is a positive number.
50. $\square$ is a negative number.
51. $-\square$ is a positive number.
52. $-\square$ is a negative number.
53. $-\triangle\triangle$ is a negative number.
54. $-\triangle\triangle$ is not a negative number.
55. $\triangle\triangle$ is a positive number.
56. $\triangle\triangle\triangle$ is a negative number.

We suggest that you make a large chart summarizing the truth-tables for compound statements, and that this chart be displayed in your classroom for ready reference. Of course, each student should keep a copy of the summary in his textbook. Suggested format:

TRUTH - TABLES FOR COMPOUND STATEMENTS						
(Components)		Conditionals	Denials	Conjunctions	Alternations	Biconditionals
p	q	if p then q	not p	p and q	p or q	p if and only if q
T	T	T	F	T	T	T
F	T	T	T	F	T	F
T	F	F		F	T	F
F	F	T		T	F	T

The truth-table for biconditionals can be derived from the truth-tables for conditionals and conjunctions.

p	q	$p \Longleftrightarrow q$
T	T	T
F	T	F
T	F	F
F	F	T

As an example of how the table is derived, consider the second line. Since p is false and q is true, $p \Longrightarrow q$ is true and $q \Longrightarrow p$ is false. $p \Longleftrightarrow q$ is thus a conjunction with first component true and second, false. Hence, $p \Longleftrightarrow q$ is false.

Since biconditionals are conjunctions, the valid inferences which one can use in connection with biconditionals are merely special cases of the valid inferences for conjunctions:

$$\frac{p \Longleftrightarrow q}{p \Longrightarrow q}, \quad \frac{p \Longleftrightarrow q}{q \Longrightarrow p}, \quad \text{and:} \quad \frac{p \Longrightarrow q \quad q \Longrightarrow p}{p \Longleftrightarrow q}.$$

GENERALIZATIONS

T. C. 35A ff. of Unit 2 contains a short description of how valid inferences can be made in connection with universal and existential generalizations. We postpone until a later Commentary a more detailed description.

* * *

(continued on T. C. 8V)

For if p is false then since $\frac{p \,\&\, q}{p}$ is valid, p & q must be false [(ii')]. Similarly, if q is false, p & q must be false. And, finally, if p and q are both true, then, since $\frac{p \quad q}{p \,\&\, q}$ is valid, p & q must be true [(ii)].

ALTERNATIONS

Next, consider alternations. Here we accept as valid:

$$\frac{p \vee q \quad p \Longrightarrow r \quad q \Longrightarrow r}{r}, \quad \frac{p}{p \vee q}, \quad \text{and: } \frac{q}{p \vee q}\,.$$

Clearly, if p is true then so is p $\vee$ q, and if q is true then so is p $\vee$ q. But if p and q are both false, then since

$$\frac{p \vee q \quad p \Longrightarrow p \quad q \Longrightarrow p}{p}$$

is valid, and p is false, while p $\Longrightarrow$ p and q $\Longrightarrow$ p are true, p $\vee$ q must be false.

p	q	p $\vee$ q
T	T	T
F	T	T
T	F	T
F	F	F

BICONDITIONALS

Finally, consider biconditionals. A biconditional, p $\Longleftrightarrow$ q, is an abbreviation for the conjunction of two conditionals.

$$p \Longrightarrow q \quad \& \quad q \Longrightarrow p\,.$$

(continued on T. C. 8U)

For, suppose p is true and q is false. Then $p \Longrightarrow q$ is false. So, by criterion (ii′), $\sim q \Longrightarrow \sim p$ is false. But this is possible only if $\sim q$ is true and $\sim p$ is false. Since, by criterion (i), there are false statements, the above argument shows that, for each true statement p, $\sim p$ is false (first line). And since we have seen that there are true statements [for example, $q \Longrightarrow (p \Longrightarrow q)$, for all statements, p and q], the same argument shows that, for each false statement q, $\sim q$ is true (second line). [Again we can check to see that this specification, as to the truth or falsity of denials, satisfies our criteria. Criteria (i) and (iii) are not disturbed. Criterion (ii) is affected only in so far as an inference,

$$\frac{\sim q \Longrightarrow \sim p}{p \Longrightarrow q},$$

occurs in a deduction. But, if $\sim q \Longrightarrow \sim p$ is true, then either $\sim q$ is false or $\sim p$ is true; by the truth-table we are considering for denials, either q is true or p is false, so $p \Longrightarrow q$ is true.]

CONJUNCTIONS

Next, consider conjunctions. Here we accept as valid inferences of the kinds suggested by:

$$\frac{p \,\&\, q}{p}, \qquad \frac{p \,\&\, q}{q}, \qquad \text{and:} \quad \frac{p \quad q}{p \,\&\, q}.$$

These lead to the truth-table:

p	q	p & q
T	T	T
F	T	F
T	F	F
F	F	F

(continued on T. C. 8T)

then anything inferred from true statements by the use of modus ponens or conditionalizing (or by any combination of inferences of these two kinds) is true. The question of whether this is necessarily the case if discharge of premisses occurs is somewhat more complicated. But consider, for example, a deduction in which only one premiss, say p, is discharged, and that this happens as the last step in the deduction. We wish to make sure that if the remaining premisses are true then the conclusion, say $p \Longrightarrow q$, is also true. Now, if p is true then, since q has been deduced from true premisses by modus ponens and conditionalizing, q is true and, by the first line of the truth-table, $p \Longrightarrow q$ is true. On the other hand, if p is false, then, whether q is true or false, $p \Longrightarrow q$ is true. This argument can be extended to a general, inductive proof that each consequence of true premisses is true, if the only logical rules used are modus ponens and conditionalization with the associated discharge of premisses. So criterion (ii) is satisfied. Finally, criterion (iii) is certainly satisfied.]

DENIALS

Let us next take up the case of denial statements. One kind of inference which we shall accept as valid is that indicated by:

$$\frac{\sim q \Longrightarrow \sim p}{p \Longrightarrow q} \ .$$

That is, one can infer a conditional from its contrapositive.

This new rule leads us to the following truth-table for denials.

p	$\sim$ p
T	F
F	T

(continued on T. C. 8S)

The first line, for example, tells us that a conditional statement is true when both of its components are true; the third line tells us that a conditional statement is false when its antecedent is true and its consequent is false. That the results summarized in this truth-table must hold for any definition of 'true' which satisfies the criteria (i) - (iii) may be seen in the following manner. Since, for all statements p and q, p $\Longrightarrow$ q is a consequence of q [conditionalizing], it follows from (ii) that p $\Longrightarrow$ q must be true if q is true. Hence, the first two lines of the table. Since q is a consequence of p and p $\Longrightarrow$ q, it follows from (ii′) that if q is false, then either p or p $\Longrightarrow$ q must be false. Hence, the third line of the table [for if q is false then, if p is true, the other premiss, p $\Longrightarrow$ q, must be false]. Finally, if p and q are false, then p $\Longrightarrow$ q must be true. For if p and q are false and p $\Longrightarrow$ q is false then, by (iii), every conditional with a false antecedent and a false consequent is false. In particular, the conditional q $\Longrightarrow$ (p $\Longrightarrow$ q) is false when p and q are false. But q $\Longrightarrow$ (p $\Longrightarrow$ q) is a tautology. And, each tautology is true, because, by (ii′), if a false statement is a consequence of a set of statements, then one of these statements must be false. But a tautology is a consequence of the empty set, and the empty set contains no false statements. So, if p and q are false, p $\Longrightarrow$ q must be true. Hence, the fourth line of the truth table. [T. C. 50A ff. of Unit 2r contains a brief discussion of the notion of truth in connection with conditionals.]

[We have still to check that the specifications summarized in the table are consistent with our three criteria. Criterion (i) is certainly not contradicted. As to criterion (ii), it is clear that if the truth of conditional statements is determined by appeal to the truth-table

(continued on T. C. 8R)

helpful to agree on some criteria which a definition of 'true' should satisfy. In the first place, if such a definition is to indicate a difference among statements, it should be such that

(i) Not every statement is true.

In the second place, it should be such that

(ii) Each consequence of true statements is true.

An equivalent formulation of this second criterion is:

(ii′) If some consequence of a set of statements [premisses] is false, then at least one of the premisses is false.

Besides these criteria, there is a third which is more controversial:

(iii) Whether a compound statement is true should be determined by which of its components are true.

CONDITIONALS

We shall now see that these three criteria together with the logical rules stated above [concerning modus ponens and conditionalizing] dictate the following "truth-table" for conditional statements.

p	q	$p \Longrightarrow q$
T	T	T
F	T	T
T	F	F
F	F	T

(continued on T. C. 8Q)

[Note: The fact that on conditionalizing one can discharge a premiss is sometimes indicated by saying:

$$\frac{\begin{matrix}[p]\\ q\end{matrix}}{p \Longrightarrow q} \quad \text{is valid.]}$$

Example 3. It is sometimes the case that a statement is a consequence of the empty set of premisses [for example: If John is rich then John is rich]. Such statements are often said to be "true on logical grounds alone", or to be tautologies. The following diagram shows that for all statements p and q, $q \Longrightarrow (p \Longrightarrow q)$ is a tautology.

$$\frac{\dfrac{\overset{*}{q}}{p \Longrightarrow q}}{q \Longrightarrow (p \Longrightarrow q)}\ *$$

The two steps are both examples of conditionalizing. The only premiss is q, and the '*'s call to your attention the fact that this premiss is discharged by the second (and last) inference. [Note that although $p \Longrightarrow q$ depends on q (first inference), $q \Longrightarrow (p \Longrightarrow q)$ depends on no premiss at all.]

TRUTH

Later we shall discuss other kinds of valid inferences which involve denials, alternations, conjunctions, and biconditionals. But now we want to make some remarks concerning the notion of truth. For some purposes it is desirable to be able to label statements 'true' or 'false' (i.e., 'not true'). This is not the place for an exhaustive commentary on Pilate's question 'What is truth?', but it will be

(continued on T. C. 8P)

and r is the statement 'John knows how to solve quadratic equations', then the preceding inference may be abbreviated as:

$$\frac{p \Longrightarrow q \qquad q \Longrightarrow r}{p \Longrightarrow r} \; .$$

The following diagram indicates how the inference from the premisses $p \Longrightarrow q$ and $q \Longrightarrow r$ to the conclusion $p \Longrightarrow r$ can be validated by a series of inferences each of which is either of the modus ponens kind or of the conditionalizing kind.

$$\cfrac{\cfrac{\cfrac{\overset{*}{p} \qquad p \Longrightarrow q}{q} \qquad q \Longrightarrow r}{r}}{p \Longrightarrow r}\;*$$

The first inference, from p and $p \Longrightarrow q$ to q, is by modus ponens, as is the second, from q and $q \Longrightarrow r$ to r. At this stage, one sees that r is a consequence of p, $p \Longrightarrow q$, and $q \Longrightarrow r$. The final step of inferring $p \Longrightarrow r$ from r is an example of conditionalizing. As remarked above in connection with this kind of inference, since $\underline{r}$ follows from $p \Longrightarrow q$, $q \Longrightarrow r$, and $\underline{p}$, we know that $\underline{p \Longrightarrow r}$ follows from $p \Longrightarrow q$ and $q \Longrightarrow r$. That is:

$$\frac{p \Longrightarrow q \qquad q \Longrightarrow r}{p \Longrightarrow r}$$

is valid. In our diagram we have noted the fact that the conclusion of the last inference does not depend on the premiss p by writing a '*' at the end of the line which we drew to symbolize "drawing" the last inference, and writing another '*' over the premiss p which this inference "discharged".

(continued on T. C. 8O)

The rule of logic according to which inferences of this kind are valid is called modus ponens. An example of this type of inference is inferring the statement '$5 \cdot 2 = (4 + 1) \cdot 2$' from the conditional 'if $5 = 4 + 1$ then $5 \cdot 2 = (4 + 1) \cdot 2$' and its antecedent '$5 = 4 + 1$'.

Another type of valid inference is symbolized by:

$$\frac{q}{p \Longrightarrow q} .$$

This kind of inference is sometimes called conditionalizing. Its importance lies chiefly in the fact that if one is able to infer a statement q from a statement p together with other statements then, on conditionalizing, one can conclude that $p \Longrightarrow q$ is a consequence of the other statements alone. We give examples of this.

Example 1. If we agree that 'John is cold' is a consequence of two statements 'John is outside' and 'The outside temperature is -40°, then, by what has just been said about conditionalizing, we should further agree that 'If John is outside then John is cold' is a consequence of the one statement 'The outside temperature is -40°'.

Example 2. Still a third type of inference, the hypothetical syllogism, can be validated on the basis of the rules of logic thus far discussed. An example of a hypothetical syllogism is inferring the statement 'If John is a high school graduate then he knows how to solve quadratic equations' from the statements 'If John is a high school graduate then he has studied mathematics' and 'If John has studied mathematics then he knows how to solve quadratic equations'. If p is the statement 'John is a high school graduate', q is the statement 'John has studied mathematics',

(continued on T. C. 8N)

generalization by dropping the quantifier, and replacing each occurrence of the variable that was generalized by a name for a member of the domain of the variable, is called <u>an instance of the generalization.</u> Thus, for example, '$2 < 2 + 1$' is an instance of (6), and '$2\pi = 3$' is an instance of (7). [Statement (6) has other instances which are sentences, but not statements. For example: $y - 2 < (y - 2) + 1$.] [In examples (1) - (5), the components of these compound statements were simple statements, and in (6) and (7) the sentences which were generalized were simple sentences. This restriction is unasserted. For example:

$$\sim (1 = 0) \Longrightarrow (7 > 9 \vee 9 > 7)$$

is a conditional statement whose antecedent is a denial and whose consequent is an alternation; and:

$$\forall_x (\exists_y (x < y))$$

is a universal generalization of a sentence which is itself an existential generalization of the simple sentence '$x < y$'. The statement '$\forall_x(\exists_y (x < y))$' asserts that, for every x, there is a y such that $x < y$. In short, it asserts that there is no largest directed number.]

<u>VALID INFERENCES</u>

For mathematics, it is important that there are logical connections, among statements, which hold because of the <u>forms</u> of the statements involved. For example, if p and q are statements, then q is a consequence of the two statements p and $p \Longrightarrow q$ [or: one can infer q from p and $p \Longrightarrow q$]. A simple way to indicate this kind of inference is:

$$\frac{p \quad p \Longrightarrow q}{q}\ .$$

(continued on T. C. 8M)

These seven examples of compound statements are, in order, a conditional, a denial, an alternation, a conjunction, a biconditional, a universal generalization, and an existential generalization. It will be convenient to rewrite these as:

(1) $1 = 0 \implies 2 < 5$

(2) $\sim (1 = 0)$

(3) $7 > 9 \lor 9 > 7$

(4) $2 < 5 \ \& \ 5 + 3 = 4 \cdot 2$

(5) $3 = 2 + 1 \iff 3 + 7 = (2 + 1) + 7$

(6) $\forall_x (x < x + 1)$

(7) $\exists_x (2x = 3)$

The symbols '$\implies$', '$\sim$', '$\lor$', '&', and '$\iff$' are called <u>sentence connectives</u>; '$\forall_x$' is a <u>universal quantifier</u>, and '$\exists_x$' is an <u>existential quantifier</u>. The statements '$1 = 0$' and '$2 < 5$' are the components of the conditional (1) and, more particularly, '$1 = 0$' is the <u>antecedent</u> of (1) and '$2 < 5$' is the <u>consequent</u> of (1). Statement (2) has the single component '$1 = 0$'. Statement (3) has '$7 > 9$' as its first component and '$9 > 7$' as its second component. Statement (4) has '$2 < 5$' as its first component and '$5 + 3 = 4 \cdot 2$' as its second component. Statement (5) has '$3 = 2 + 1$' as its first component and '$3 + 7 = (2 + 1) + 7$' as its second component. Statement (6) is obtained by universally generalizing the variable 'x' in the sentence '$x < x + 1$', and (7) is obtained by existentially generalizing the variable 'x' in the sentence '$2x = 3$'. [In FIRST COURSE, our agreement is that the domain of the variable 'x' in (6) and (7) is the set of all directed numbers.] Each statement obtained from a

(continued on T. C. 8L)

question. Hence, (1) is a correct set selector for $\{0, 1\}$, and (2) is not. In other words, each of the numbers 0 and 1 satisfies (1). The student notes that '$0 = 0$ or $0 = 1$' and '$1 = 0$ or $1 = 1$' are true because one of the components of the alternation is true. He sees that (2) is not a correct set selector, since he notes that '$0 = 0$ and $0 = 1$' and '$1 = 0$ and $1 = 1$' are false because one of the components of the conjunction is false.

* * *

We present here a derivation of the truth-tables for alternations and conjunctions, and, for completeness, the truth-tables for conditionals, denials, and biconditionals. Most of this presentation is for your own information; however, if you can teach any of it to your students, it will be of help to them in SECOND COURSE.

COMPOUND STATEMENTS

It is convenient to classify the statements which occur in mathematics according to their structure. A statement may be either simple:

$$1 = 0, \quad 2 < 5, \quad 5 + 3 = 4 \cdot 2, \quad 7 > 9,$$

or compound:

if $1 = 0$ then $2 < 5$,
$1 \neq 0$ [that is, it is not the case that $1 = 0$],
$7 > 9$ or $9 > 7$,
$2 < 5$ and $5 + 3 = 4 \cdot 2$,
$3 = 2 + 1$ if and only if $3 + 7 = (2 + 1) + 7$,
for each x, $x < x + 1$,
there is an x such that $2x = 3$.

(continued on T. C. 8K)

$3 > 6$ or $3 \not> 6$ ..... true

$3 > 6$ and $3 \not> 6$ ..... false

$2 \neq 1 + 1$ or $9 \neq 3 \times 3$ ..... false

$2 \neq 1 + 1$ and $9 \neq 3 \times 3$ ..... false

The construction of a set selector for Exercise 1 will provide you with an opportunity for introducing the rule for alternations. An obvious name for the solution set of Exercise 1 is '$\{x: x(x - 1) = 0\}$', but the set selector here is not especially informative; certainly, this name is not as informative as '$\{0, 1\}$'. Students may volunteer

'$\{x: x = 0$ or $x = 1\}$' and '$\{x: x = 0$ and $x = 1\}$'

as names for the solution set. Which of the sentences:

(1) $x = 0$ or $x = 1$

and:

(2) $x = 0$ and $x = 1$

selects the set $\{0, 1\}$? Recall the "dramatic" approach to the concept of a set selector described on T. C. 5A ff. Each directed number introduces itself to the set selector and the set selector must decide whether the number satisfies it or not. [In the case of '$x + 2 = 5$', the selector must ask of each number if its sum with 2 is 5. In the case of '$x = 9$', the selector must ask of each number if it is 9.]

In the case of (1), the selector must ask of each number if it is 0 or 1. 0 and 1 are the only two numbers who answer 'yes'. In the case of (2), the selector must ask of each number if it is 0 <u>and</u> 1. Of course, there is no number which can answer 'yes' to this

(continued on T. C. 8J)

role as a connective more apparent.] There are rules for judging the truth or falsity of alternations [sentences whose two components are connected by an 'or'] and of conjunctions [two components connected by an 'and']. The need for such rules arises in writing simple set selectors in Part C; moreover, there will be continued application of these rules in Unit 4 and in later UICSM courses. We shall now give these rules, suggest how they can be made reasonable to students, and then give a logical basis for the rules.

We use the symbols and notation commonly found in logic texts: '∨' for 'or', '&' for 'and', 'T' for 'true', 'F' for 'false', and 'p' and 'q' in place of sentences.

p	q	p ∨ q	p & q
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	F

As an example of how the table is read, consider the next-to-the-last row. It tells us that if the first component of a compound sentence is false and the second component is true, then if the compound sentence is an alternation, it is true, and if it is a conjunction it is false. [Check the following examples.]

$9 + 2 = 3 + 8$ or $6 + 1 = 3 + 5$ .... true

$9 + 2 = 3 + 8$ and $6 + 1 = 3 + 5$ ...... false

(continued on T. C. 8I)

R and S. The answer is 'no' because now the first factor is converted into a name for a positive number, and the other two factors are converted into names for negative numbers, so the whole expression is converted into a name for a positive number. Continue moving your finger to the right, asking the appropriate questions as you proceed.

* * *

After students have drawn pictures of the loci of the sentences in Part C, ask them to use the brace-notation to name the solution sets. Here are our answers to these exercises.

1. $\{x:\ x = 0 \text{ or } x = 1\}$
2. $\{x:\ x < 0 \text{ or } x > 1\}$
3. $\{x:\ 0 < x < 1\}$, or: $\{x:\ x > 0 \text{ and } x < 1\}$
4. $\{x:\ x = 0 \text{ or } x = 3\}$
5. $\{x:\ x = 1 \text{ or } x = 2\}$
6. $\{x:\ x < 1 \text{ or } x > 2\}$
7. $\{x:\ x = 2 \text{ or } x = -3\}$
8. $\{x:\ x < 2 \text{ and } x > -3\}$
9. $\{x:\ x = 2 \text{ or } x = -1\}$
10. $\{x:\ x = \frac{2}{3} \text{ or } x = \frac{6}{5}\}$
11. $\{x:\ x = 1 \text{ or } x = 2 \text{ or } x = 3\}$
12. $\{x:\ x < 1 \text{ or } 2 < x < 3\}$
13. $\{x:\ x = -4 \text{ or } x - 3 \text{ or } x = -1\}$
14. $\{x:\ -4 < x < -1 \text{ or } x > 3\}$
15. $\{x:\ x = 4\}$
16. $\{x:\ x = 5 \text{ or } x = 0\}$

The job of writing simple set selectors calls for a good understanding of the distinction between the use of 'or' and the use of 'and'; 'or' and 'and' are called sentence connectives and are used in constructing compound sentences out of simple ones. [Another sentence connective is 'if . . . then ---', which could be abbreviated by '$\Longrightarrow$' to make its

(continued on T. C. 8H)

3. $\{9, 15, 17\}$ is a subset of $\{x: x > 4\}$

4. $\{x: 3 < x < 5\}$ is a subset of $\{x: 2 < x < 7\}$

5. $\{y: -1 < y \leq 5\}$ is not a subset of $\{z: -1 \leq y < 5\}$

[An abbreviation for 'is a subset of' is '$\subseteq$'. An abbreviation for 'is not a subset of' is '$\not\subseteq$'.]

6. $\{x: x = 7\} \subseteq \{y: yy = 49\}$

7. $\{x: |x| > 2\} \subseteq \{x: |x| \geq 2\}$

8. $\{a: -5 < a < 5\} \subseteq \{b: -4 < b < 4\}$

9. $\{x: x > 10\} \subseteq \{x: x \leq 100\}$

10. $\{x: x = \frac{1}{2}\} \not\subseteq \{x: |xx| = \frac{1}{4}\}$

11. $\{x: 7 \leq x \leq 7\} \subseteq \{y: 9 \leq y \leq 9\}$

12. $\{x: x = 5\} \not\subseteq \{y: y = 5\}$

13. $\{x: x + 1 = x + 2\} \subseteq \{x: x + 1 = 2\}$

[Note: A detailed explanation for handling Exercise 13 above is given on T. C. 24D.]

* * *

The exercises in Part C should be diagramed because the diagram makes it easy to analyze the problem. For example, in solving Exercise 12, one can point to locations to the left of point R and ask if replacements for coordinates of points ''out here'' will convert the expression to the left of the '<' into a name for a negative number. The answer is 'yes'. Now, put your finger on point R. The answer here is 'no'. Then put your finger on a point between

(continued on T. C. 8G)

III. True or false?

1. $\{x: \ x + 1 = 9\} = \{x: \ 3x + 2 = 26\}$
2. $\{y: \ 3y - 2 = 7\} = \{y: \ 6y + 7 = 19\}$
3. $\{a: \ 5a = 5\} = \{b: \ b - 1 = -b + 1\}$
4. $\{\square : \ 4\square + 5 = 9\} = \{\square : \ 3\square = \frac{1}{3}\}$
5. $\{x: \ 7x + 1 > 15\} = \{y: \ 6y < 12\}$
6. $\{t: \ -t > 7\} = \{x: \ x < -7\}$
7. $\{s: \ s + 3 = 12 + 3\} = \{s: \ s = 12\}$
8. $\{k: \ k + 7 < 15 + 7\} = \{k: \ k < 15\}$
9. $\{t: \ 7t = 14\} = \{t: \ \frac{2t + 1}{5t - 1} = \frac{5}{9}.\}$
10. $\{a: \ a + 1 = a\} = \{a: \ \frac{2a + 1}{2} - 1 = 0\}$

IV. If all the elements of a first set are also elements of a second set, the first set is a <u>subset</u> of the second set. For example.

$\{3, 5, 7\}$ is a subset of $\{1, 3, 4, 5, 6, 7, 12, 15\}$,
$\{x: x = 5\}$ is a subset of $\{x: xx = 25\}$,
$\{x: x < 6\}$ is a subset of $\{y: y < 9\}$,
but $\{x: x > 4\}$ is <u>not</u> a subset of $\{t: \ t > 5\}$

because at least one of the elements of $\{x: x > 4\}$ is not an element of $\{t: t > 5\}$.

True or false?

1. $\{3, 8, 9\}$ is a subset of $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$
2. $\{6, 7\}$ is a subset of $\{2, 4, 6, 8, 10, 12\}$

(continued on T. C. 8F)

Practice with the ideas and notation introduced above is provided by the following

<u>Supplementary</u> <u>Exercises</u>.

I. Which of the following expressions are names of sets and which are not ?

1. $\{y: \; y + 5 = 12\}$
2. $\{k: \; 3k - 2 = 9 + 5k\}$
3. $\{m: \; 3m + 1 > 9\}$
4. $\{x: \; 2 - 5x\}$
5. $\{s: \; \frac{2s - 7}{3} = 8\}$
6. $\{z: \; \frac{3 - 4z}{8}\}$
7. $\{x: \; 1 + \frac{9}{2}x + 5\}$
8. $\{u: \; 3 < 2u + 7 < 19\}$
9. $\{x: \; x = 1 + x\}$
10. $\{r: \; r + 9 = 9 + r\}$

II. In the name:

$$\{x: \; x + 1 > 9\}$$

the sentence '$x + 1 > 9$' is called <u>a</u> <u>set</u> <u>selector</u>. This term is appropriate because the sentence '$x + 1 > 9$' can serve to select from the set of all directed numbers just those numbers which are elements of the solution set; in other words, just those elements which satisfy '$x + 1 > 9$'. Another set selector which has the same solution set as '$x + 1 > 9$' is '$x > 8$'.

For each of the following names of sets, write a set selector which is different from the one contained in the name but which has the same solution set.

1. $\{x: \; 3x = 5\}$
2. $\{y: \; 7y \leq 21\}$
3. $\{k: \; k + 7 \leq 6\}$
4. $\{m: \; 2m + 5 < 19\}$
5. $\{a: \; |a| + 4 \leq 15\}$
6. $\{t: \; 3t + 7 = 3(t - 1)\}$

(continued on T. C. 8E)

the solution set of 'k + 10 > 10' is not as simple nor as informative as it could be, although it is technically correct. So, we are ready to ask for the simplest name for the solution set of 'k + 10 > 10', a name which would communicate all of the information about the solution set with the least amount of computing necessary by the person who sees the name. Clearly, such a simple name is: $\{k: k > 0\}$. Now, return to the exercises, and ask students to write, in similar fashion, the simplest names for the solution sets. [$\{x: x > 2\}$, $\{x: x > 5\}$, $\{x: x < 5\}$, $\{x: x > -5\}$, $\{k: k > 0\}$, $\{m: m < 3\}$, $\{\square : \square < -3\}$] The answer to Exercise 8, '$\{p: p < \frac{5}{2}\}$', is much simpler than the geometric description called for in the original instructions. Have students use this notation in naming the solution sets for the rest of the exercises.

Call their attention to the answers for Exercises 16, 17, and 19:

$\{t: t = 2\}$, $\{b: b = 2\}$, and: $\{r: r = 2\}$.

Clearly, the sentences in these three exercises have the same solution set. The fact that the three names involve different pronumerals does not change the fact that the names denote the same set. Just as one writes the true statement:

$$7 + 9 = 8 \times 2,$$

one writes the true statement:

$$\{t: t = 2\} = \{b: b = 2\},$$

and even:

$$\{x: \frac{x}{8} + 3 = 2\frac{1}{2}\} = \{z: z = -4\}.$$

(continued on T. C. 8D)

set which contains many elements? Consider Exercise 1. We can think of this sentence as selecting from the set of all directed numbers the set of those numbers each of which when added to 3 gives a sum greater than 5. In fact, each sentence can be thought of as selecting its solution set from the set of all directed numbers. Thus, sentences such as those in Part B are sometimes called set selectors. And it is possible to use the set selector itself in constructing a name (or: description) of its solution set. Here is how a mathematician does this. First, he writes a left-brace: {. This indicates that he is going to write a name of a set. Then he writes a pronumeral and a colon immediately after the brace: {x: . This indicates that the set in question consists of numbers selected from the set of all directed numbers; that is, the domain of 'x' is understood [by convention] to be the set of all directed numbers. The colon indicates that the writer is about to tell how the elements in the set are to be selected. The next part of the name is the set selector itself, followed by a right-brace to indicate that the description is complete:

$$\{x: x + 3 > 5\}.$$

[This symbol is read as 'the set of all [numbers] x such that $x + 3 > 5$'. The symbol '$\{x: \ldots\}$' is sometimes called 'a set-abstraction operator'.] Thus, the solution set in Exercise 2 is $\{x: x - 1 > 4\}$, and the solution set in Exercise 10 is $\{w: -2 < w + 1 < 5\}$. Now, if a student is asked to give (or: describe) the solution set of a sentence, say, '$k + 10 > 10$' all he needs to do is write: $\{k: k + 10 > 10\}$. Similarly, if a student is asked to give the sum of 87.3 and 92.9, all he needs to do is write: $87.3 + 92.9$. Of course, although this last answer is technically correct, it is not as simple as it could be, and for many purposes, not as informative as it could be. Similarly, the name '$\{k: k + 10 > 10\}$' for

(continued on T. C. 8C)

Answers to Part B:

1. $\overrightarrow{SL}$ 2. $\widetilde{\overrightarrow{LZ}}$ 3. $\overrightarrow{LZ}$ 4. $\overrightarrow{YL}$

5. $\overrightarrow{TL}$ 6. $\overrightarrow{PZ}$ 7. $\overrightarrow{AZ}$

8. the set of all points to the left of the point with coordinate $\frac{5}{2}$

9. $\overrightarrow{SZ}$ 10. $\overline{AQ}$ 11. $\{Q\}$ 12. $\{Z\}$

13. $\{R\}$

14. the set consisting of the point with coordinate 14

15. $\{X\}$ 16. $\{S\}$ 17. $\{S\}$ 18. $\{B\}$

19. $\{S\}$ 20. $\{X\}$ 21. $\{Z\}$ 22. $\{Z\}$

Exercises 8 and 14 point up the difficulty in giving a geometric description of the solution set or locus of a sentence. For one thing, even if a quick description of a set is possible (as in the case of the other exercises), one needs to have a labelled picture of the number line available in order to be understood. Secondly, there are many sets whose geometric descriptions are hard to give (as in the case of Exercises 8 and 14). Is there an easy way to describe a set without using a picture?

Students have already learned that there is such a way in the case of a set which contains just a few <u>elements</u> [bring this word into the discussion]. For example, the set consisting of just the two numbers, 9 and -15, is $\{9, -15\}$. Similarly, the solution set in Exercise 11 is $\{4\}$. But how about an easy and non-geometric way of describing a

(continued on T. C. 8B)

B. Describe the locus of each of the following. (Use the diagram below.)

1. $x + 3 > 5$
2. $x - 1 > 4$
3. $x - 1 < 4$
4. $x + 3 > -2$
5. $k + 10 > 10$
6. $2m < 6$
7. $-9 > 3\square$
8. $2p < 5$
9. $-2 < -v$
10. $-2 < w + 1 < 5$
11. $x + 5 = 9$
12. $x + 4 = -2$
13. $a - 3 = -2$
14. $t - 7 = 7$
15. $-3t = 12$
16. $-7t = -14$
17. $3b + 1 = 7$
18. $5s - 3 = -13$
19. $\frac{2r + 6}{2} = 5$
20. $\frac{x}{8} + 3 = 2\frac{1}{2}$
21. $\frac{2}{3}x = -4$
22. $\frac{2x}{3} = -4$

-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
Z	Y	X	A	B	C	T	R	S	P	Q	L

*C. Describe the locus of each of the following. (Use the diagram above.)

1. $x(x - 1) = 0$
2. $x(x - 1) > 0$
3. $x(x - 1) < 0$
4. $x(x - 3) = 0$
5. $(x - 1)(x - 2) = 0$
6. $(x - 1)(x - 2) > 0$
7. $(x - 2)(x + 3) = 0$
8. $(x - 2)(x + 3) < 0$
9. $(2x - 4)(5x + 5) = 0$
10. $(3x - 2)(6 - 5x) = 0$
11. $(x - 1)(x - 2)(x - 3) = 0$
12. $(x - 1)(x - 2)(x - 3) < 0$
13. $(x + 4)(x - 3)(x + 1) = 0$
14. $(x + 4)(x - 3)(x + 1) > 0$
15. $(x - 3) + (x - 4) + (x - 5) = 0$
16. $(x - 3)(x - 2) = 6$

We want a bit of plausible reasoning from the student. Here is an example of the kind of reasoning we would be happy to observe:

> Any number which is the root of this equation must be such that its sum with 3 is 8. Suppose there were a different root. It must be either greater than or less than 5. If it were greater than 5, then its sum with 3 would be greater than 8. If it were less than 5, then its sum with 3 would be less than 8.

If a student questions the statement:

> If it were greater than 5, then its sum with 3 would be greater than 8,

you can argue as follows:

> Suppose you have a number greater than 5. This means that there is a $k > 0$ such that the number is $5 + k$. Now, if you add this number to 3, you obtain $(5 + k) + 3$ or $8 + k$ which (since $k \neq 0$) means that the sum is not 8.

In giving answers orally, do not permit students to say things such as 'x is 5' or 'y is -2'. Although a case can be made in defense of such statements ["For every x, x + 3 is 8 if and only if x is 5."], they are not direct answers to the question, 'What are the roots of ...'. Such statements tend to make students think they are "looking for x" or "trying to find what x is". This is the kind of "general-numberish" or "literal-numberish" point of view which we have taken pains to avoid. If a student says 'x is 5', respond with 'I know what 'x' is; it's the twenty-fourth letter in the alphabet. I want to know what the roots are.'.

* * *

Some students may recall mechanical rules for solving equations, rules which they learned in an earlier grade. They will tend to stop using these rules when you ask only for roots [no work to be shown, please] and put a premium on speed in giving answers.

* * *

When a student gives an incorrect answer, the only way to correct him is to replace the pronumeral in the equation by a name for the alleged root. For example, if a student says '2' for Exercise 2, you can respond by saying 'Does 2 + 17 = 15?'.

* * *

After the student has solved quite a few of the exercises in Part A, you should return to Exercise 1 and raise the following question:

How do you know that 5 is the only root of this equation?

Naturally, the student cannot give a rigorous proof that 5 is the only root, but that is not sufficient reason to avoid this important question.

(continued on T. C. 9C)

Refer to T. C. 4A regarding this use of the word 'expression'. It is better to use the word 'sentence' here in every case in which 'expression' occurs.

* * *

Use set-language to discuss the points raised in the introductory paragraph on page 3-9. For example, the elements in the solution set of an equation are the roots of the equation. Elements in the solution set of a sentence are sometimes called <u>solutions of the sentence</u>.

* * *

The equations in Part A are so-called one-step equations . You will find that students can solve these equations without any prior instruction other than that given in the discussion at the top of page 3-9. Do not present such things as "equation axioms" or any other formal device for solving equations. Don't talk about doing things to both sides, or compare with balances, etc. Note that the equations in Exercises 9, 10, and 11 need to be solved just as those in the other exercises.

* * *

In giving the roots of these equations, students should write only numerals for the roots. They should not state their answers in the form 'x = ...'. If they do this, point out that the instructions ask for numbers as answers rather than for equations. Although it is the case [Exercise 1] that the root of 'x = 5' is precisely the root of 'x + 3 = 8', and that the root of 'x = 5' is more obvious, the student's job is one of finding the root, not of finding a simpler equation that has the same root as the given equation. On the other hand, if the job was that of finding the solution set of 'x + 3 = 8' then either '{5}' or '{x: x = 5}' is an acceptable answer.

(continued on T. C. 9B)

3.03 Equations.--In the exercises of the preceding section you were searching for numbers which satisfy expressions like '$x + 5 = 7$', '$3a = 8$', and '$y = 6$'. Such expressions are called equations. Expressions like '$x > 5$' and '$3x + 2 < 4$' are called inequalities. The numbers which satisfy an equation are called roots of the equation or, simply, roots. Thus, the equation:

$$5t - 7 = 3$$

has the number 2 as a root because 2 satisfies the equation, that is, because

$$\text{'}5(2) - 7 = 3\text{'}$$

is a true statement. When we have found all of the roots of an equation, we say that we have solved the equation. (Sometimes the roots are called solutions of an equation.)

EXERCISES

A. Solve these equations.

1. $x + 3 = 8$	2. $y + 17 = 15$
3. $a + 4 = 0$	4. $t + 12 = 9$
5. $9 + m = 10.4$	6. $18 + p = 8.6$
7. $\square + 2 = 28$	8. $12 + \triangle = 1$
9. $x = -1$	10. $y = 1.5$
11. $\square = 17$	12. $-\square = 5$
13. $x - 9 = 7$	14. $y - 4 = 3$
15. $A - 2 = -3$	16. $K - 9 = 0$
17. $3 - x = 1$	18. $26 - y = 18$
19. $4 - q = 6$	20. $-5 - t = -8$
21. $5\triangle = 15$	22. $3\square = 12$
23. $9x = 16$	24. $8y = -24$
25. $3s = 17$	26. $2r = -9$
27. $-4\triangle = 0$	28. $-3\square = +27$

(continued on next page)

Then, direct his attention to the '12m'. Ask about the numeral that could be written in place of 'm' so that the resulting expression would be a name for -24. This should be enough help. If the student needs help on another equation of the same type, say, Exercise 21, tell him to review the thinking he did in Exercise 10.

It may be necessary to ask the students to refrain from getting help from students in other classes or from their parents. They should understand that the purpose of these exercises is to get them to develop their own methods for solving equations. They will be operating at a disadvantage later on if they use a rule someone has given them. Students should feel very confident of themselves in solving equations of the type shown before they attack Part C, which begins on page 3-11.

Exercises 37 through 44 of Part A are difficult for some students, and should be supplemented by the following exercises.

1. $\frac{5}{7}x = 10$ 2. $\frac{5x}{7} = 10$ 3. $\frac{5}{7x} = 10$

4. $\frac{2}{5}y = 6$ 5. $\frac{2y}{5} = 6$ 6. $\frac{2}{5y} = 6$

7. $\frac{8}{z} = 16$ 8. $\frac{8}{-z} = 16$ 9. $\frac{8}{3z} = 24$

10. $\frac{7}{x} = 1$ 11. $\frac{7}{2x} = 1$ 12. $\frac{7}{-x} = 1$

13. $\frac{9}{2y} = \frac{9}{2}$ 14. $\frac{18}{7x} = \frac{18}{7}$ 15. $\frac{19}{6y} = \frac{19}{3}$

16. $\frac{8}{3 - x} = 4$ 17. $\frac{8}{x - 3} = 4$ 18. $\frac{8}{|3 - x|} = 4$

* * *

Part B contains so-called two-step and three-step equations. No formal instruction is necessary for these exercises. Students should work many of these exercises in class. Give individual help only when asked [or when a student is making consistent errors], and make this help intuitive only. By 'intuitive help' we mean help such as the following:

> Suppose a student is having trouble with Exercise 10. You might place your thumb over '12m' and ask him what numeral could be written in place of '12m' so that the resulting statement would be true. Let him think about this until he replies that '-24' would "work"., It is not necessary to ask him how he obtained '-24'.

(continued on T. C. 10B)

29. $\square \div 8 = 5$

30. $\Delta \div (-2) = 3$

31. $\frac{x}{5} = 2$

32. $\frac{y}{4} = -2$

33. $\frac{t}{17} = 0$

34. $\frac{y}{-3} = 5$

35. $\frac{-\square}{8} = 1$

36. $-\frac{\Delta}{6} = 1$

37. $\frac{1}{2}\square = 4$

38. $\frac{3}{5}\Delta = 6$

39. $\frac{5}{3}t = 15$

40. $\frac{7}{2}z = -35$

41. $\frac{3}{\square} = 1$

42. $\frac{6}{\Delta} = 2$

43. $\frac{21}{x} = 7$

44. $\frac{8}{y} = 24$

45. $5 = 8 + x$

46. $20 = y - 10$

47. $y - 17.8 = 25.2$

48. $z + 97.6 = 18.3$

49. $12x = 93$

50. $18z = 38$

51. $8.5k = 17.3$

52. $x - 22\frac{1}{2} = 48\frac{3}{4}$

53. $xx = 100$

54. $tt = -81$

55. $2AA = 98$

56. $yy - 12 = 24$

B. Find the roots of the following equations.

1. $3x = 9$

2. $3x + 4 = 16$

3. $2x - 5 = 1$

4. $7x + 8 = 43$

5. $5 + 8x = 29$

6. $2 + 3q = 29$

7. $5x + 4 = -11$

8. $3y + 12 = -12$

9. $7k + 9 = 2$

10. $15 + 12m = -9$

11. $2x - 12 = -16$

12. $7 + 3x = 7$

13. $5 - 2x = 1$

14. $9 - 2x = 13$

15. $5x + 7 = 18$

16. $3 + 4x = 9$

17. $6y - 5 = 17$

18. $9z - 8 = 5$

19. $\frac{1}{2}x - 7 = 12$

20. $\frac{2}{3}y + 2 = 16$

21. $\frac{2}{5}x - 8 = -2$

22. $\frac{1}{8}k - 1 = 1$

(continued on next page)

* * *

Emphasize that students check by replacing the occurrences of the pronumeral in the given equation. Also, stress that checking is just to catch possible computational errors.

11. $\underline{\{x: 8xx - 2x(3 + 4x) = -12\}}$

$\{x: 8xx - 6x - 8xx = -12\}$

$\{x: 2x - 8xx = -12\}$

$\{x: 8xx - 8xx - 6x = -12\}$

$\{x: -6x = -12\}$

* * *

Take plenty of time with the samples in Part C. Write the given equation above the "simplified" equation:

(1) $5x - 3 + 4x + 5 = 20$

(2) $9x + 2 = 20$

Then ask the following question:

> Do you think that any number which is a root of equation (1) must also be a root of equation (2)?

Make sure that students can justify their answer to this question [equivalent algebraic expressions]. Now ask this question:

> Do you think that any number which is a root of equation (2) must also be a root of equation (1)?

The student must also be able to justify his answer to this question.

The idea of asking these two questions about a pair of equations should become very familiar to the student. You will need to consider these questions again later in the unit when we undertake the development of the concept of pairs of equivalent equations. Naturally, the practice he has had in dealing with equal solution sets of sentences will help him considerably in the work on equivalent equations.

(continued on T. C. 11E)

1. $\underline{\{x:\ 2x + 3x + 7 = 17\}}$

$\{x:\ 2x + 10 = 17\}$
$\{x:\ 5x + 7 = 17\}$
$\{x:\ 12x = 17\}$

2. $\underline{\{r:\ 5r + 3 - 2r - 12 = 12\}}$

$\{r:\ 5r + r - 12 = 2\}$
$\{r:\ 3r - 9 = 12\}$
$\{r:\ 5r - 11 = 2\}$

3. $\underline{\{x:\ 3x + 4(2x + 7) = 83\}}$

$\{x:\ 7x + 2x + 7 = 83\}$
$\{x:\ 3x + 8x + 7 = 83\}$
$\{x:\ 3x + 8x + 28 = 83\}$
$\{x:\ 11x + 28 = 83\}$

4. $\underline{\{k:\ 2(k - 3) + 7(k + 1) = 46\}}$

$\{k:\ 2k - 6 + 7k \div 1 = 46\}$
$\{k:\ 2k - 6 + 7k + 7 = 46\}$
$\{k:\ 9k - 13 = 46\}$
$\{k:\ 9k + 1 = 46\}$

5. $\underline{\{y:\ 5y - 3 + 4y + 5 = 20\}}$

$\{y:\ 2y + 4y + 5 = 20\}$
$\{y:\ 5y + y + 5 = 20\}$
$\{y:\ 5y - 7y + 5 = 20\}$
$\{y:\ 9y - 3 + 5 = 20\}$
$\{y:\ 9y + 2 = 20\}$

6. $\underline{\{t:\ 3(t - 4) - 5(3 - 2t) = 38\}}$

$\{t:\ 3t - 4 - 15 - 10t = 38\}$
$\{t:\ 3t - 4 - 15 + 10t = 38\}$
$\{t:\ 3t - 19 + 10t = 38\}$
$\{t:\ 3t - 9t = 38\}$
$\{t:\ 13t - 27 = 38\}$

7. $\underline{\{y:\ 6y - 5(2 - 3y) = 32\}}$

$\{y:\ y - 2 - 3y = 32\}$
$\{y:\ 6y - 10 - 15y = 32\}$
$\{y:\ 6y - 10 + 15y = 32\}$
$\{y:\ 21y - 10 = 32\}$

8. $\underline{\{k:\ 5k - 2(k - 3) - 5(4 - 5k) = 144\}}$

$\{k:\ 5k - 2k - 3 - 20 - 5k = 144\}$
$\{k:\ 5k - 2k + 6 - 20 + 25k = 144\}$
$\{k:\ 5k + (-2)k + 25k + 6 + (-20) = 144\}$
$\{k:\ 28k - 14 = 144\}$

9. $\underline{\{z:\ 8z + 3 + 2z + 9 = -28\}}$

$\{x:\ 8x + 2x + 3 + 9 = -28\}$
$\{a:\ 10a + 12a = -28\}$
$\{k:\ 10k + 12 = -28\}$

10. $\underline{\{y:\ 8(9 - 3y) + 5(6 - 7y) - 2y = 15\}}$

$\{y:\ 72 - 3y + 30 - 7y - 2y = 15\}$
$\{y:\ 72 - 24y + 30 - 35y - 2y = 15\}$
$\{y:\ 102 - 61y = 15\}$

(continued on T. C. 11D)

16. $7k + 4 = -73$	17. $3 - \frac{1}{2}x = 0$	18. $6 - y = -1$
19. $5 + 5y = 5$	20. $8 - 4t = 5$	21. $9 = 7 - 2k$
22. $\frac{1}{2}y + 1 = -\frac{1}{2}$	23. $\frac{1}{3} = \frac{2}{5} - k$	24. $9 + \frac{k}{10} = 11$
25. $\frac{1}{2}(3 + 4x) = \frac{3}{2}$	26. $\frac{1}{3}(s + 6) = 22$	27. $\frac{2}{5}(3 - x) = 5$
28. $\frac{5 + 3x}{4} = 20$	29. $\frac{8 - 3y}{7} = 2$	30. $1 = \frac{-2x - 7}{5}$
31. $8 - 2m = 5$	32. $9 + 5z = 0$	33. $16 + 4f = -16$
34. $1 + 3k = 5$	35. $2 - 5s = 4$	36. $1 + \frac{3}{2}t = 25$
37. $2x + 2 = 10$	38. $6 - 3y = 9$	39. $28 + 7x = 14$
37a. $2(x + 1) = 10$	38a. $3(2 - y) = 9$	39a. $7(4 + x) = 14$
40. $2(x + 1) = 13$	41. $3(2 - y) = 17$	42. $7(4 + x) = 23$
43. $3(x + 1) = 3x$	44. $2(x - 5) = 2$	45. $3(2 + x) = 3x + 6$
46. $\lvert 4 + x \rvert = 12$	47. $\lvert 5 - 2x \rvert = 5$	48. $\lvert 8 + 5x \rvert = 18$
49. $3 + \lvert x - 2 \rvert = 8$	50. $8 - \lvert 3 - x \rvert = 13$	51. $\lvert 3 - x \rvert - 8 = 13$

* * *

The work in Part C will be expedited, and the set-language taught earlier will be reviewed if students do the following exercises <u>before</u> they start Part C.

<u>Exploration Exercises</u>

In each exercise you are given a name for a set with a line drawn under the name. Below the line there are other names for sets. Draw a loop around those names which stand for the same set as the one given above the line.

(continued on T. C. 11C)

Do not fail to point out the similarity between the equations in Exercises 23 and 25 and those in Exercises 24 and 26. Ask students if the equation in Exercise 23 has the same roots as the equation '$\frac{t + 6}{2} = 9$'. Try to elicit from them the statement that they could tell this without solving the equations by noticing that '$\frac{1}{2}(t + 6)$' and '$\frac{t + 6}{2}$' are equivalent algebraic expressions. A bit of review on what is meant by 'equivalent algebraic expressions' would fit in well here because Part C requires a good understanding of this term.

* * *

Mr. Marston and his class enjoyed playing with the equation:

$$\frac{3 + |x|}{8 + |x|} = \frac{2}{3}$$

and variations of it obtained by replacing the '$\frac{2}{3}$' in it by a '$\frac{1}{2}$', a '$\frac{3}{4}$', a '$\frac{5}{6}$', and a '$\frac{7}{8}$'.

* * *

Note that each of the equations in Exercises 41, 42, 43, 45, and 46 has two roots. The equation in Exercise 44 has no roots.

<u>Supplementary Exercises for Part B</u>.

1. $7a - 4 = 17$	2. $3 + 2x = 19$	3. $5 - 6y = -7$
4. $-x - 6 = -10$	5. $\frac{1}{2}x + 9 = 11$	6. $7y - 1 = -50$
7. $8t + 4 = 28$	8. $3 + 9y = -6$	9. $3 - 9y = 6$
10. $2 - k = -6$	11. $\frac{1}{3}y - 17 = 9$	12. $6x - 2 = 2$
13. $8y + 4 = 4$	14. $6x - 6 = 12$	15. $8 = 2y - 3$

(continued on T. C. 11B)

23. $\frac{1}{2}(t + 6) = 9$

24. $\frac{1}{3}(s - 4) = 2$

25. $\frac{x + 2}{5} = 3$

26. $\frac{y + 7}{2} = 9$

27. $\frac{m - 5}{8} = 3$

28. $\frac{n - 2}{4} = 6$

29. $\frac{4 - \square}{7} = 2$

30. $\frac{8 - \Delta}{10} = 8$

31. $\frac{2a + 7}{3} = 5$

32. $\frac{3b - 7}{4} = 8$

33. $\frac{3t + 5}{4} = -1$

34. $\frac{5z - 4}{-3} = -2$

35. $2\frac{1}{2}x + 1\frac{3}{4} = 3$

36. $3\frac{1}{4}y - 7\frac{1}{5} = 8\frac{4}{5}$

37. $5.2y - 4.3 = 8.6$

38. $9.2b - 3.8 = -5.7$

39. $7.8 - 3.9z = 0$

40. $8.7 - 5.3u = 5.2$

41. $|x - 5| = 5$

42. $|3 - 2y| = 1$

43. $2 + |3 - 3x| = 8$

44. $5 + |7 - 4x| = 2$

45. $2ZZ - 3 = 15$

46. $9kk - 3 = -2$

C. Solve these equations.

Sample 1. $5x - 3 + 4x + 5 = 20$

Solution. The expression on the left side of the equal sign can be simplified by combining terms, a process you learned in Unit 2. We know that for every x,

$$5x - 3 + 4x + 5 = 9x + 2$$

Since '$5x - 3 + 4x + 5$' and '$9x + 2$' are equivalent algebraic expressions, we know that when we replace each 'x' in both expressions by a numeral for the same number, we get two expressions for the same number. Now, in view of this fact, if we replace '$5x - 3 + 4x + 5$' in the given equation by '$9x + 2$', we obtain a new equation:

$$9x + 2 = 20$$

and we know that this equation and the given equation must have the same roots. Explain.

(continued on next page)

Solve the new equation. It is like those you solved in Part B. It has the root 2. Does the given equation have the root 2? Check to see.

$$\begin{aligned} & 5(2) - 3 + 4(2) + 5 \\ = {} & 10 - 3 + 8 + 5 \\ = {} & 20 \end{aligned}$$

Yes, the given equation also has the root 2.

<u>Sample</u> <u>2</u>. $3(x - 4) - 5(3 - 2x) = 38$

<u>Solution</u>. Applying the methods you learned in Unit 2 for simplifying algebraic expressions we know that for every x,

$$\begin{aligned} & 3(x - 4) - 5(3 - 2x) \\ = {} & 3x - 12 + [-5(3 - 2x)] \\ = {} & 3x - 12 + [-15 - (-10x)] \\ = {} & 3x - 12 + [-15 + 10x] \\ = {} & 3x - 12 - 15 + 10x \\ = {} & 13x - 27 \end{aligned}$$

Now, to solve the given equation, all we need to do is to solve the equation:

$$13x - 27 = 38$$

This equation is like those you have solved before. It has the root 5. Therefore, the given equation has the root 5.

It is a good practice to check that the given equation is satisfied by 5 because we could have made errors in simplifying the long algebraic expression. A check in that case is likely to show that an error has been made.

$$\begin{aligned} & 3(5 - 4) - 5[3 - 2(5)] \\ = {} & 3(1) - 5[3 - 10] \\ = {} & 3 - 5(-7) \\ = {} & 3 + 35 \\ = {} & 38 \end{aligned}$$

Supplementary Exercises for Part C.

Solve these equations.

1. $3x - 5 + 8x - 12 = 5$

2. $6m - 4 - 4m + 8 = 4$

3. $7A - 6 + 13A + 18 = 8$

4. $9b - 21 - 3b - 2 = 1$

5. $3(n - 2) + 5 - 6n = 17$

6. $9x + 7 - 2(x - 7) = -21$

7. $2(6k - 3) + 4(5k - 4) = 6$

8. $a - (6 - 2a) = -2$

9. $2b - 2(3b - 7) = 2$

10. $5(7r - 1) - 6(7r - 1) = 15$

11. $y - (9 - y) - (8 - y) = -8$

12. $13(r - 4) + 5(3r - 3) = -11$

13. $12 - 3(s + 5) - 2(9s - 7) = 95$

14. $6(2 - 9y) + 4(2y - 9) + y = -114$

15. $18m - (m + 5) - 3(m - 2) + 4(m - 5) = -55$

16. $7(x - 2) - 8x - 3 = 100$

17. $9(x - 1) - 9(x + 1) + 2(x - 2) - 2(2 + 2x) = 0$

18. $x(3 - 2x) - 5x(x - 3) + 7x(x + 2) - 18x = 42$

19. $10(10 + yy) - 9(yy + 2) - 2(1 + yy) = -1$

20. $2xx + 2(6x - 3) + 6(2x - 1) - x(2x + 3) = -5$

21. $3(x - 5) - 3x = 6$

22. $11(2 + s) + 2(11 - s) + 2 - 11s = 4$

23. $5 + 4(5 - 2r) - 6r - 3(5r + 2) + r = 103$

24. $7 + 8(x - 4) = 15$

25. $(5n - 2) - 7 - (n - 2) + 7n - 5(2n - 5n) = 6$

26. $7rr + 5r(r - 1) - 12r(5 + r) - 100 = 30$

27. $t - (1 - t) + (t - 1) - 1 = 0$

28. $2w + 2w(w - 2) + 2(2 - ww) + 4w = 4$

29. $7x - 34(2x - 5) - 9(x - 7) - 64x = -1442$

30. $m - 12m(m - 3) + 16m(3/4m - 11) + 175m = 252$

1. $7y - 2 + 5y = 10$
2. $8A - 3 - 2A = 15$
3. $5z + 4 - 3z + 2 = 2$
4. $m + 2m + 1 + 4m = 1$
5. $3x + 2(x - 2) = 11$
6. $3y + 5(4 - y) = 26$
7. $4(B - 6) + 3(2B + 1) = -41$
8. $8(7 - k) + 12(3 + 2k) = 12$
9. $5(2r - 3) - 3(r + 7) = -15$
10. $7(3 - 5s) - 2(2s - 4) = -89$
11. $2a - 2(3a - 1) = 22$
12. $3b - 5(2 - 4b) = -10$
13. $x - (2 - x) = 36$
14. $3y - (12 - 2y) = 3$
15. $x - (x - 1) - (x - 2) = -2$
16. $5m - 2(m - 3) - 5(4 - 5m) = 144$
17. $\frac{1}{2}(2x - 6) + \frac{1}{3}(3x + 9) = 16$
18. $\frac{1}{3}(9x + 12) + \frac{1}{5}(5x - 15) = 25$
19. $\frac{2}{5}(x - 3) + \frac{3}{5}(4 + x) = 5\frac{1}{5}$
20. $\frac{3}{4}(8x + 9) - \frac{5}{2}(9 - 4x) = \frac{1}{4}$
21. $5 - 7(2 - x) + 4(2x - 5) = 31$
22. $8(9 - 3y) + 5(6 - 7y) - 2y = 15$
23. $5(3 - 2z) - 6(5z - 2) + 8(3z - 5) = -21$
24. $5.3(4 + 3m) - 8.2(2m - 1) = 7.9$
25. $10.4(5.3 - 2.1p) - 11.6(7.3p - 6.5) = 43.9$
26. $4x + 2(5 - 2x) = 10$
27. $2x - 2(x - 3) = 7$
28. $x(x - 5) + 2x(3 - x) + xx = 5$
29. $3y(2 - y) - 4y(3 - y) + 6y - 5 = 116$
30. $6(a - 3) + 7(a - 3) - 8(a - 3) - 4(a - 3) = 7$
31. $5(bb - 3) - 8(bb - 3) + 3(bb - 3) = 6$

In some classes it will be necessary to teach the common mensurational formulas almost from scratch. Many eighth grade textbooks have large sections on this topic with adequate developments and exercises.

* * *

Note that the letters in '$P = 2(L + W)$' are pronumerals, not abbreviations. [The letters used serve as mnemonic devices.] To use the formula you put a numeral for the measure of the length in place of 'L', and a numeral for the measure of the width in place of 'W' [a common unit of measure for both dimensions]; the resulting expression on the right of '=' is a numeral for the measure of the perimeter.

3.04 Formulas. --In earlier grades you used formulas to solve problems. For example, suppose you are given the following problem:

> A farmer wants to build a fence around a field which has a rectangular shape. If the field is 270 feet long and 112 feet wide, how many feet of fencing does he need?

It is easy to see that in order to solve this problem you need to find the perimeter of a rectangle. You learned a rule like:

To find the perimeter of any rectangle, add its length and its width and double this sum.

You may also have learned a formula which tells you the same thing in a more concise way:

For a rectangle, $P = 2(L + W)$.

Note that the formula contains the pronumerals 'P', 'L', and 'W'. The formula tells you that for every L and W, if L and W are measures of the length and the width of a rectangle, then 2(L + W) is the measure of its perimeter (<u>provided that all quantities are expressed in the same unit</u>--inches, or feet, or yards, etc.).

Now let us use the formula to solve the given problem. We replace 'L' by '270' and 'W' by '112', and obtain the equation:

$$P = 2(270 + 112)$$

To solve our problem we need to solve this equation, that is, we need to find a number such that when a name for this number replaces 'P' in the equation, we get a true statement. You can see at once that the required

number is 2(270 + 112). A simpler name for this number (and a name which would be more useful to the farmer who wants to build the fence) is '764'. So, the rectangle's perimeter is 764 feet and the farmer needs 764 feet of fencing.

Here is another problem for which you use the same formula:

> Milton wants to bend a piece of thin copper tubing to form a rectangle which will be 4 inches wide. If the piece of tubing is 22 inches long, how long will the rectangle be?

This problem involves a rectangle, its dimensions, and its perimeter. You know one of the dimensions, the width; you know the perimeter, and you want to find the other dimension, the length. So, in the equation:

$$P = 2(L + W)$$

replace 'P' by '22' and 'W' by '4'. You get the equation:

$$22 = 2(L + 4).$$

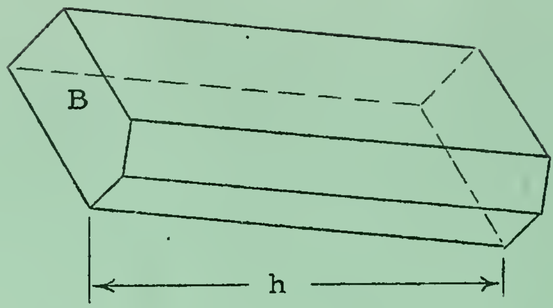
To solve this equation you find a replacement for 'L' which gives a true statement. 7 satisfies the equation. Therefore, Milton should make his rectangle 7 inches long.

EXERCISES

A. What does each of the following formulas tell you?

Sample. For a quadrilateral, $P = a + b + c + d$.

Solution. The formula tells you that for every a, b, c, and d, if a, b, c, and d are measures of the lengths of the four sides of a quadrilateral, then $a + b + c + d$ is a measure of the quadrilateral's perimeter.



is hB. This rule applies to all prismatic figures (right and oblique) such as polygonal prisms and cylinders.

* * *

It is well to point out to the students the similarity between the formula for the volume of a pyramid and the formula for the volume of a cone. In each case, it is $\frac{1}{3}$ times the measure of the area of the base times the measure of the height. In the case of the pyramid, the base-area measure is indicated by the 'B' of the formula, while in the case of the cone, it is indicated by the 'πrr' of the formula.

* * *

The familiar checking device for selecting formulas should be taught:

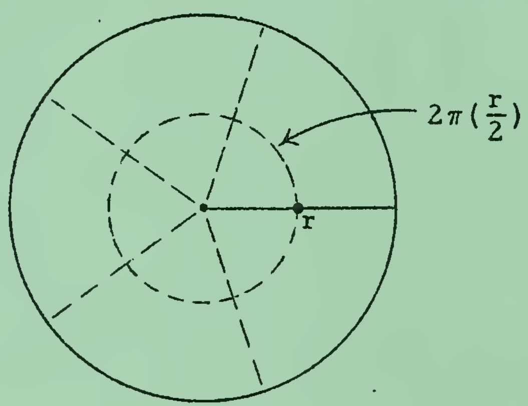
perimeter	$\longrightarrow$	adding measures of the lengths
area	$\longrightarrow$	multiplying measure of length by measure of length
volume	$\longrightarrow$	multiplying measure of area by measure of length

* * *

In Part B do not make the use of formulas in solving problems a burdensome task for students by insisting that they follow the three steps we have indicated for each problem they attempt. It is quite likely that for certain problems [very easy ones], the steps really annoy the student. If you want students to follow the steps in most of the problems tell them that you are interested not only in whether they are able to solve the problems but also in whether they can organize their work so that other people can follow it.

$h\left(\frac{b + b'}{2}\right)$, or $\frac{1}{2}h(b + b')$. In the case of a triangle, the measure of the starting length is b and the measure of the final length is 0, so the measure of the area is $h\left(\frac{b + 0}{2}\right)$, or $\frac{1}{2}hb$.

In the case of a circle and its interior, the figure can be thought of as being generated by a line segment which is rotated around one of its end-points. Here the generating segment does not "change" in length, but the distance moved by the segment is taken as the distance moved

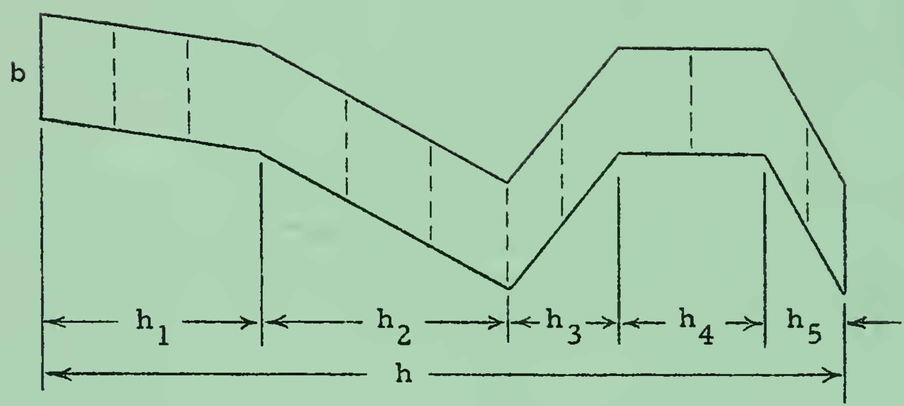


by its mid-point. So, if the measure of the line segment's length is r and the distance moved is $2\pi(\frac{r}{2})$, then the measure of the area is $[2\pi(\frac{r}{2})] \times r$, or πrr.

Volume formulas

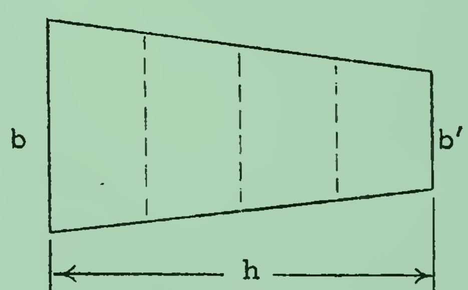
A three-dimensional figure and its interior can be viewed as being generated by a two-dimensional figure and its interior moving parallel to its original position. If the measure of the area of the generating figure is B and the distance moved is h, then the measure of the volume

(continued on T. C. 16G)



$$
\begin{aligned}
A &= bh_1 + bh_2 + bh_3 + bh_4 + bh_5 \\
&= b(h_1 + h_2 + h_3 + h_4 + h_5) \\
&= bh \\
&= hb
\end{aligned}
$$

A figure such as a trapezoid and its interior can be viewed as being swept out by a line segment which moves parallel to its original position and whose length "changes" at a uniform rate. If the measure of the starting length is b, and the measure of the final length is b′, and the

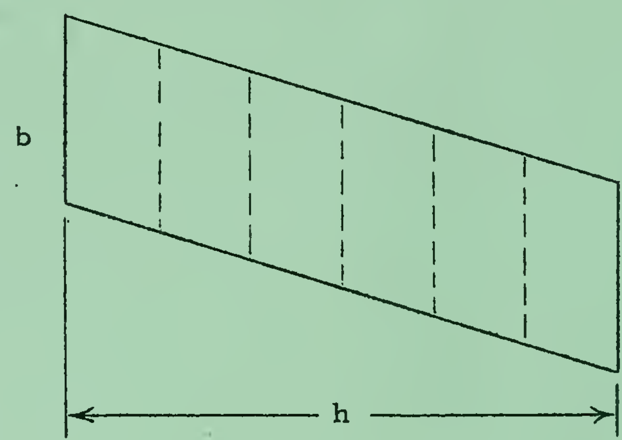


distance moved is h, then the area is h times the measure of the <u>average</u> length of the generating segment; that is, the measure of the area is

(continued on T. C. 16F)

Here is another method of developing some of the area formulas.

One can think, quite intuitively, of certain plane figures and their interiors as being generated by a moving line segment. For example, a parallelogram and its interior is swept out by a line segment which moves parallel to its original position.

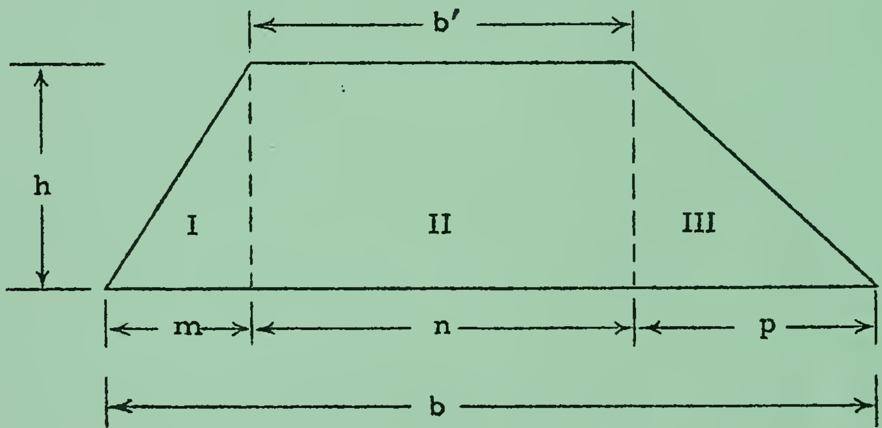


If the measure of the line segment's length is b [base of the parallelogram] and the distance moved is h [height of the parallelogram], then the measure of the area is hb. This rule holds for a figure such as the following.

(continued on T. C. 16E)

Area measure of $\triangle AGD = \frac{1}{2}(b + n)h$, and area measure of ▭ $CDGH = bh$. So, area measure of ⏢ $AHCD = bh + \frac{1}{2}(b + n)h$. Area measure of $\triangle BHC = \frac{1}{2}(b + n)h$ = area measure of $\triangle AGD$. But, area measure of ▱ $ABCD$ = area measure of ⏢ $AHCD$ minus area measure of $\triangle BHC$; therefore, area measure of ▱ $ABCD = bh$.

You may consider a trapezoid in this way:



measure of the area of the trapezoid = measure of the area$_{\text{I}}$ + measure of the area$_{\text{II}}$ + measure of the area$_{\text{III}}$

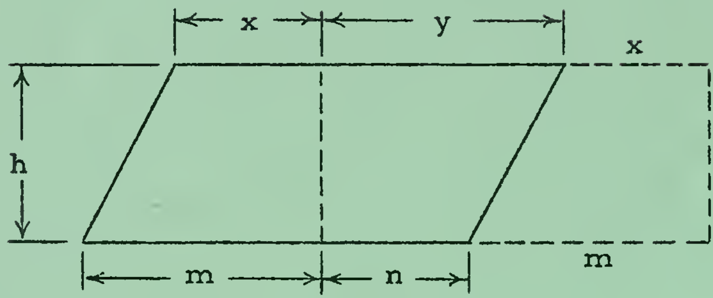
or, measure of the area of the trapezoid

$$
\begin{aligned}
&= \frac{1}{2}mh + nh + \frac{1}{2}ph \\
&= \frac{1}{2}mh + \frac{1}{2}(2nh) + \frac{1}{2}ph \\
&= \frac{1}{2}h(m + 2n + p) \\
&= \frac{1}{2}h(m + n + b' + p) \\
&= \frac{1}{2}h(b + b').
\end{aligned}
$$

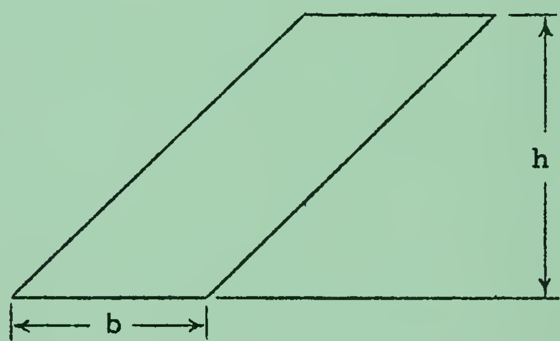
* * *

(continued on T. C. 16D)

Here is another picture you could use:

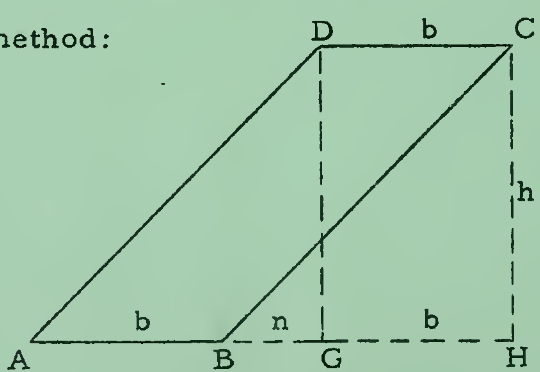


You may want your more able students to consider this case [with base and height being those so indicated]:



and ask whether they can obtain the measure of the area of the parallelogram by using the formulas they know for finding the measures of the areas of rectangles and triangles.

This is one method:



(continued on T. C. 16C)

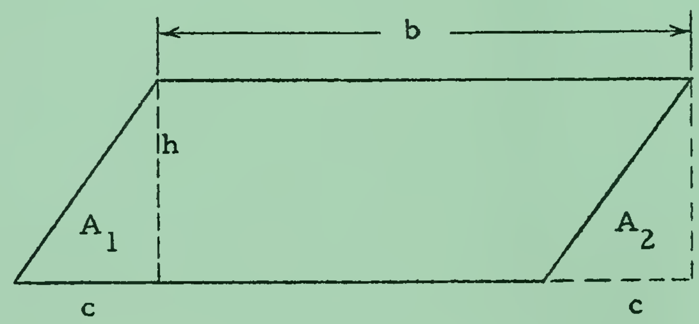
The pronumeral 'B' is sometimes used in volume formulas as a space-holder for numerals which name the areas of bases. It is also used in Exercise 7. Point out to students that 'b′' is sometimes used in that formula instead of 'B'.

* * *

There are several interesting ways to develop many of these formulas, and to help students remember them.

<u>Area formulas</u>

Consider the parallelogram. You may draw on the board a picture such as this:



and help the students discover that the portion of the parallelogram [that is, the portion of the <u>interior</u> of the parallelogram] which is labelled 'A_1' would "fit" the figure labelled 'A_2', thus giving a rectangle, the measure of whose base is b and the measure of whose height is h.

(continued on T. C. 16B)

1. For a rectangle, $A = LW$.
2. For a square, $A = ss$.
3. For a square, $P = 4s$.
4. For a triangle, $P = a + b + c$.
5. For a triangle, $A = \frac{1}{2}hb$.
6. For a parallelogram, $A = hb$.
7. For a trapezoid, $A = \frac{1}{2}h(b + B)$.
8. For a circle, $A = \pi rr$.
9. For a circle, $C = 2\pi r$.
10. For a rectangular solid, $S = 2(LW + WH + HL)$.
11. For a rectangular solid, $V = LWH$.
12. For a cube, $S = 6ee$.
13. For a cube, $V = eee$.
14. For a prism, $V = Bh$.
15. For a pyramid, $V = \frac{1}{3}Bh$.
16. For a circular cylinder, $V = \pi rrh$.
17. For a circular cone, $V = \frac{1}{3}\pi rrh$.
18. For a sphere, $S = 4\pi rr$.
19. For a sphere, $V = \frac{4}{3}\pi rrr$.
20. For every right triangle, $cc = aa + bb$.
21. For every triangle, $A + B + C = 180$.

<u>B</u>. Solve each of the following problems. To give yourself practice using formulas, solve each problem by

(1) selecting the appropriate formula,

(2) substituting numerals for all but one of the pronumerals, and

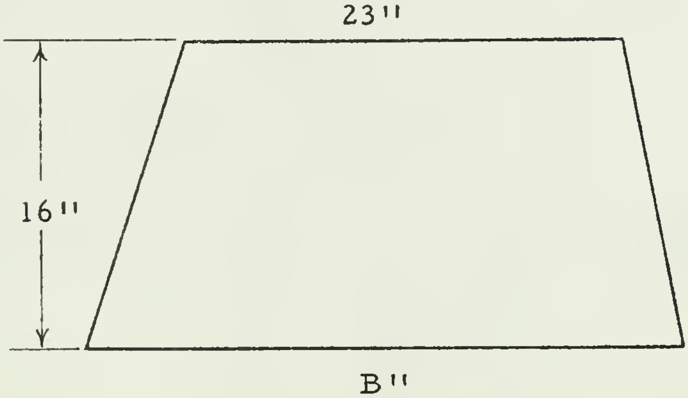
(3) solving the resulting equation.

<u>Make a rough estimate first</u>.

Put some kind of premium on making a rough estimate before attempting a problem. Perhaps you should require that the estimated answer be written on the homework paper. There is no standard technique for making estimates. Students should work out methods of their own for estimating and you should encourage them to discuss the ways in which they make estimates. Students can often be helped by listening to their classmates' descriptions of estimating procedures. You, too, should demonstrate how you make estimates. Often, the making of an estimate gives a valuable clue to how to solve a problem. This advantage will be more noticeable in the work with 'worded' problems later in the unit.

Be sure that students estimate crudely enough to get results. Students have a tendency to "cling" to too much accuracy when they begin estimating.

Sample 1. The number of square inches in the area of the trapezoid pictured below is 432. If the length of one base is 23'', find the length of the longer base.



Solution. First, roughly estimate the length of the other base. If the trapezoid were a rectangle 23" by 16", its area would be a little less than 25×16 or 400 square inches. (It is easy to multiply 25×16: think $\frac{1}{4} \times 16 \times 100$.) Therefore, the other base is the longer one and it is somewhat longer than 25". You might guess 28" or 30".

Now, select a formula which is concerned with the area of a trapezoid. This is the formula in Exercise 7 of Part A. In the equation:

$$A = \frac{1}{2}h(b + B)$$

substitute '432' for 'A', '16' for 'h', and '23' for 'b'. We get the following equation:

$$432 = \frac{1}{2}(16)(23 + B).$$

We find that 31 is a root of this equation. Therefore, the longer base is 31 inches long. Our estimate is close enough to this answer to lead us to believe we have solved the problem correctly.

The symbol '$\overset{a}{=}$' was suggested by a student in one of Miss Wandke's classes. Since there is no standard symbol for the purpose, we shall use this one. The 'a' in the symbol reminds the student of the word 'approximately'. You may want to show the students some other symbols that are used for this purpose. Examples: $\approx$, and: $\doteq$.

* * *

Ask students to compare the surface area measure of two rectangular solids if the dimensions of one are twice the dimensions of the other. Three times? Four times? [You may have to use a few examples before students can give answers readily.] Vary the way in which you ask this kind of question. For example:

> What happens to the surface area [or: area measure] of a rectangular solid if you double the dimensions?

and:

> What happens ... if each dimension is increased by 100% of itself?

[Some of the expressions used in these questions are colloquialisms, but the student will have no difficulty in understanding them.] Also, ask similar questions about the volume of a rectangular solid. Include questions such as:

> What happens to the volume [or: volume measure] of a rectangular solid if you double two of its dimensions and triple the third?

and:

> What happens ... if you leave one dimension alone, double another, and halve the third?

Sample 2. Find the radius of a circle if the number of inches in the circumference is 72.

Solution. To estimate our answer we use the fact that $\pi \overset{a}{=} 3$. (Read '$\overset{a}{=}$' as 'is approximately equal to'.) The formula (in Exercise 9) tells us that the circumference is approximately equal to 6 times the radius. Therefore, we estimate the radius to be $\frac{1}{6}$ of 72" or 12". (Since $\pi > 3$, we know the radius is actually less than 12.)

In the equation:

$$C = 2\pi r$$

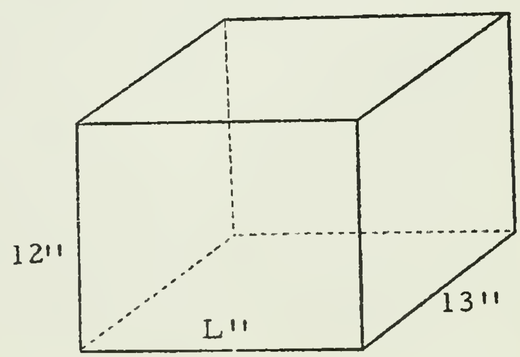
we substitute '72' for 'C' and obtain:

$$72 = 2\pi r.$$

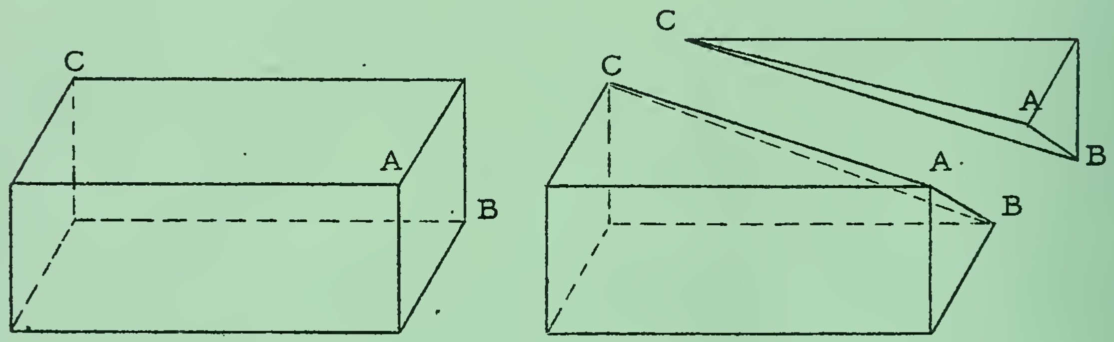
(Note: 'π' is a numeral, not a pronumeral. Explain.) We solve this last equation obtaining $\frac{36}{\pi}$ as a root. Therefore, the radius is $\frac{36}{\pi}$ inches.

The symbol '$\frac{36}{\pi}$' is the simplest name for the number $\frac{36}{\pi}$. However, if you want to use your answer to draw a circle, you will need to use another number which is "close to" $\frac{36}{\pi}$. To find a number close to $\frac{36}{\pi}$ you use a number close to π. Depending upon the precision with which you are going to make your drawing, use 3, or 3.1, or 3.14, or 3.142, etc. as such an approximation for π. Using these approximations successively for π gives 12", 11.6", 11.46", 11.458", etc. as approximations for the radius. Our estimate leads us to accept these results.

1. The area of the surface of the rectangular solid is 531 square inches. One of its dimensions is 12 inches and another is 13 inches. Find its third dimension.

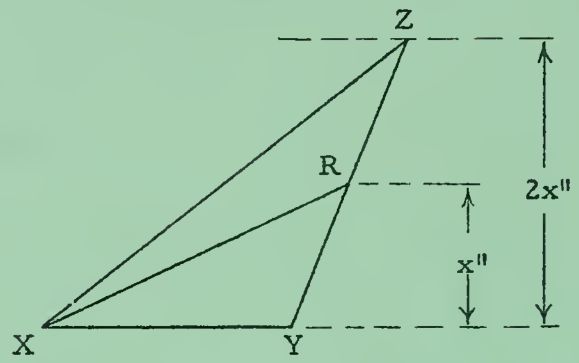


Pass a plane through the vertices A, B, and C of a rectangular solid, thereby cutting the solid into two figures. What do you call the



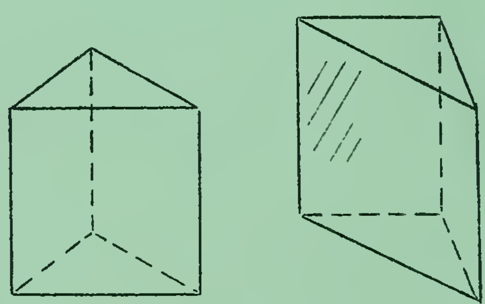
smaller figure ['triangular pyramid']? Compare its volume with that of the rectangular solid. Compare its volume with that of the other figure.

If the area of $\triangle XYZ$ is 30 square inches, what is the area of $\triangle XYR$? Of $\triangle XRZ$?



Exercise 5.

What do you call the two solids you get when you slice a cube by passing a plane through two diagonally opposite edges? How do



Triangular prisms

you compute the volume of each of these figures? Compare the volume of each with the volume of the cube. [Consider the same kind of problem with a rectangular solid cut by a plane which is passed through two diagonally opposite edges.]

(continued on T. C. 19C)

Exercise 2.

What happens to the area of the trapezoid if you double the distance between bases [that is, double the height] and leave the lengths of the bases unchanged? If you double the height and double each base length? [Ask these kinds of questions whenever appropriate as you discuss the figures on pages 3-19, 3-20, and 3-20A.]

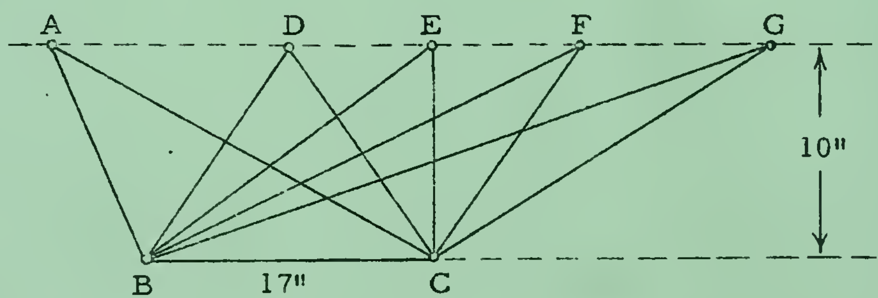
* * *

In the remainder of the Commentary for pages 3-19 through 3-21, you will find examples of variations of the problems given in the text. These may be introduced into class discussion by the teacher, or copied on ditto masters and reproduced, to be given to the students as supplementary work. These will help students gain insight into the properties of various geometric figures.

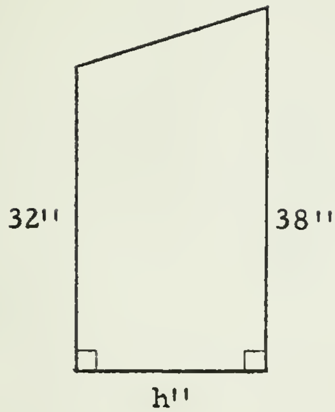
* * *

Exercise 3.

What is the area of each of the triangles ABC, DBC, EBC, FBC, and GBC?

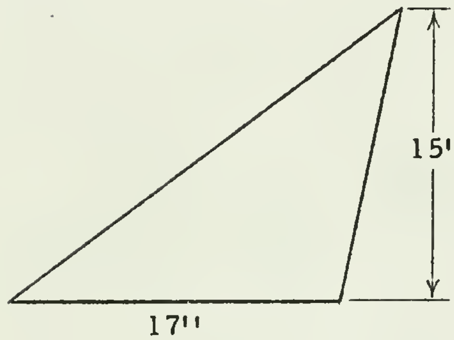


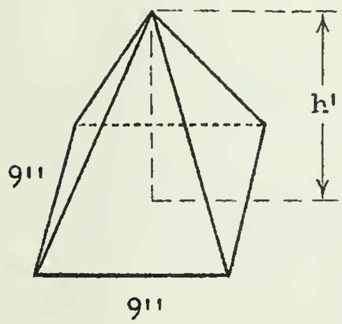
(continued on T. C. 19B)



2. The area of the trapezoid is 700 square inches. What is the distance between its bases?

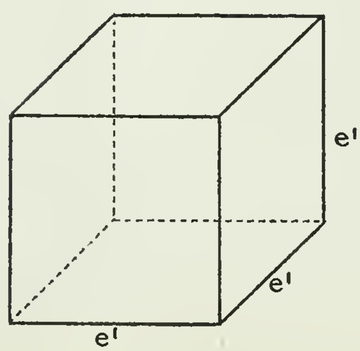
3. Find the area of the triangle.





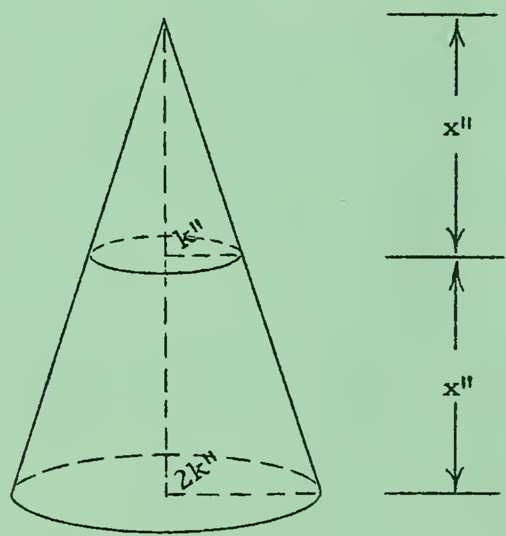
4. The volume of the pyramid is 216 cubic inches. It has a square base. Find its height.

5. The cube's volume is 8 cubic feet. What is the length of one of its edges?



Exercise 9.

At a distance halfway from the apex [vertex] of a cone to the plane of its base, cut through the cone with a plane parallel to the plane of its base. Compare the volumes of the resulting figures [a cone, and a frustum of a cone] with each other and with the volume of the original cone.

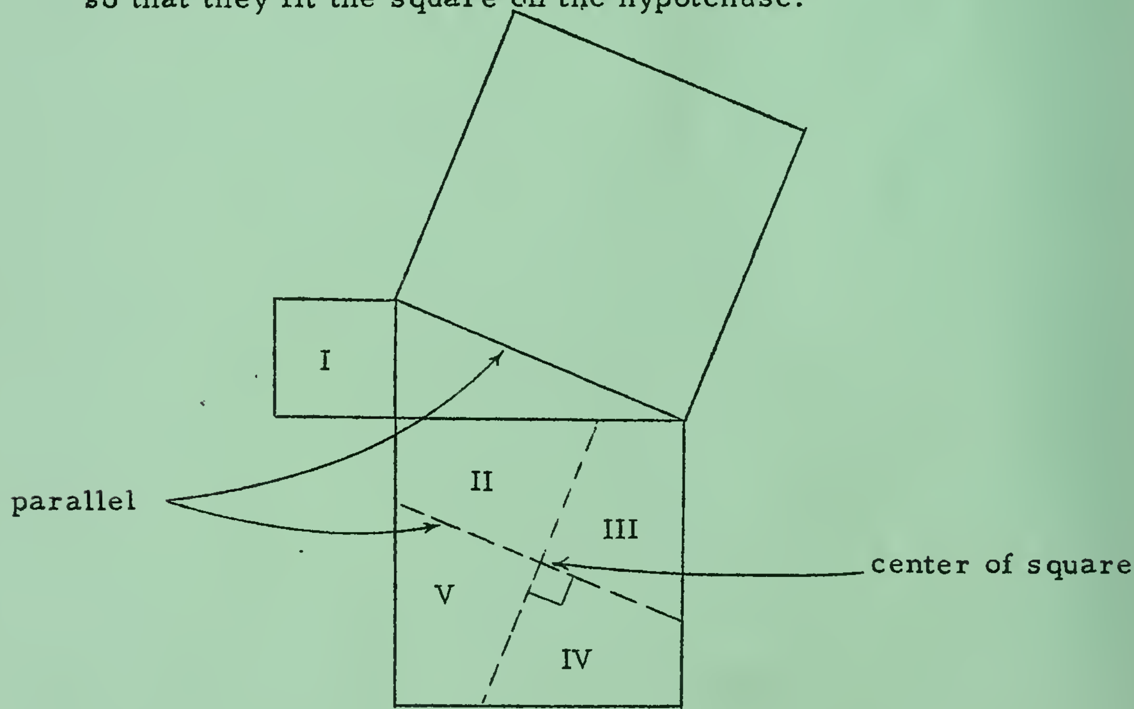


Exercise 8.

Double (triple, quadruple) lengths of legs. What is the change in the length of the hypotenuse? [Construct other problems dealing with 3-4-5 and 7-24-25 triangles.]

An interesting homework exercise which will serve to drive home the Pythagorean rule is the following.

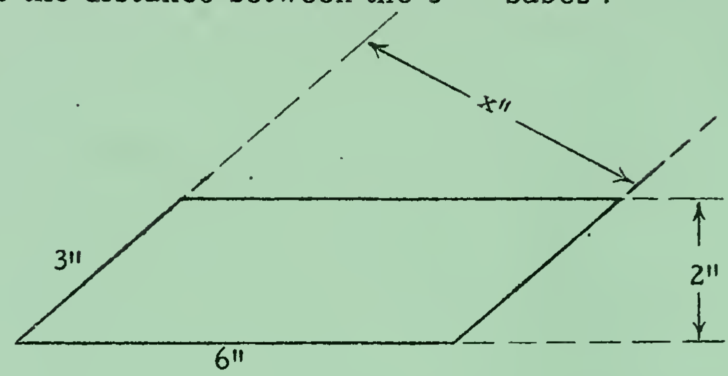
> Draw a right triangle on a piece of cardboard and draw squares on the three sides. Draw the dotted line segments as shown. Cut out the pieces I, II, III, IV, and V, and reassemble these five pieces so that they fit the square on the hypotenuse.



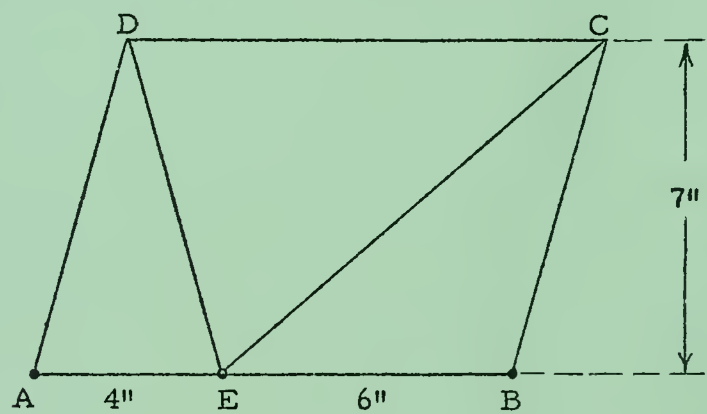
(continued on T. C. 20C)

Exercise 6.

What is the distance between the 3" - bases?



What are the areas of parallelogram ABCD, ΔAED, ΔEBC, and ΔEDC? Repeat when the length of $\overline{AE}$ is 3" and the length of

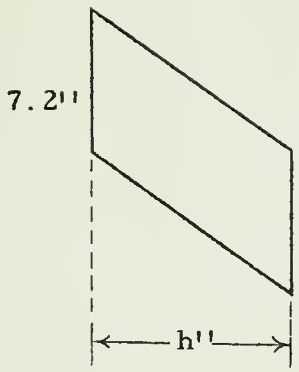


$\overline{EB}$ is 7". 2" and 8". Etc.

Exercise 7.

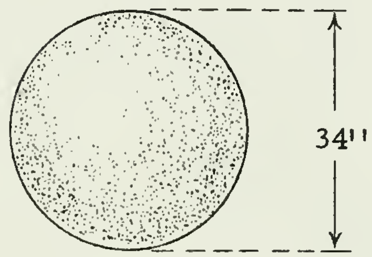
Compare the amounts of juice in two oranges if the diameter of the first orange is twice the diameter of the second orange. [Roughly 8 times as much juice in the larger.]

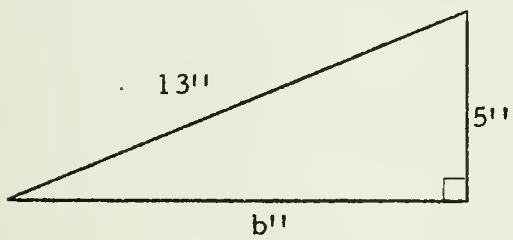
(continued on T. C. 20B)



6. What is the distance between its two parallel sides if the area of the parallelogram is 78 square inches?

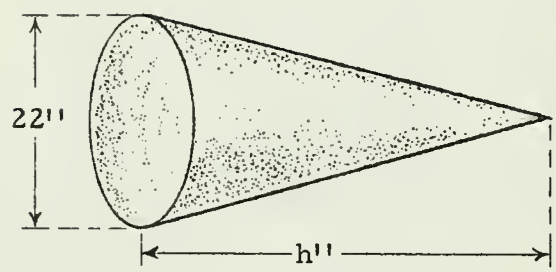
7. What is the volume of a sphere whose diameter is 34 inches?

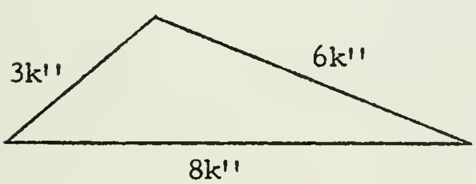




8. Find the length of the side of the right triangle.

9. Find the height of the circular cone whose volume is 50 cubic inches.





10. The perimeter of the triangle is 51 inches. Find the length of each side.

Exercise 11 requires finding the square root of 92. We are assuming that many of the students learned in the eighth grade how to find square roots. If you find that enough students have forgotten how to do this [or never learned it] to make a class presentation worth the time, then teach the method of approximating-dividing-averaging-dividing-. . . . If you are not familiar with this method, you will find it demonstrated in almost any eighth or ninth grade textbook. Undoubtedly, there will be some students who insist [or whose parents insist] on learning the old algorithm for square root extraction. You might make an after-school or extra-credit assignment out of this. Several elementary algebra books contain a detailed description and rationalization of this algorithm.

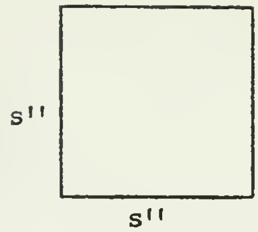
The books listed below are just a few of those which give an explanation of the method of approximating-dividing-averaging-dividing-. . . .

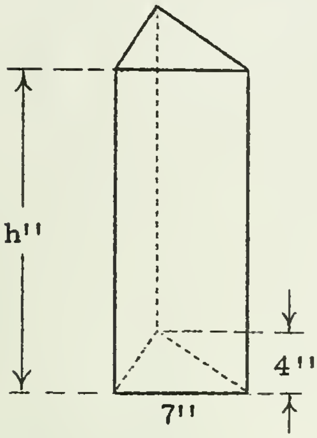
Stokes and Sanford, First Course in Algebra (New York: Henry Holt and Co., 1935) pp. 370, 371.

Virgil S. Mallory, First Algebra (Chicago: Benj. Sanborn and Co., 1950) p. 377.

Freilich, Berman, Johnson, Algebra for Problem Solving, Book 1 (Chicago: Houghton Mifflin Co., 1952) pp. 428, 429.

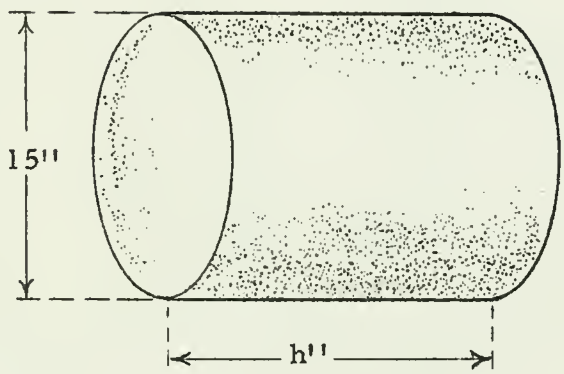
11. The area of the square is 92 square inches. Find the length of one of its sides.

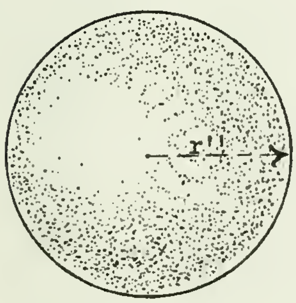




12. The volume of the triangular prism is 240 cubic inches. What is its height?

13. The volume of the circular cylinder is 1.8 cubic feet. How long is it?





14. What is the radius of a sphere whose surface has an area of 64π square inches?

Exercise 15.

Introduce the formula 'i = prt', and consider questions such as:

> What is the change in interest if the principal is doubled, the annual rate is halved, and the time is unchanged?

Bring in one or two easy compound interest problems in order to contrast simple interest with compound interest.

* * *

Explain in as much detail as necessary such terms as:

profit, overhead, gross profit, net profit, margin, cost price, selling price, commission, commission rate, discount, [Exercise 19], discount rate, and list price.

Note: In many of the problems which follow you will have to make up your own formula.

15. Find the simple interest on a loan of \$750 borrowed for 2 years at an annual rate of 5.5%.

16. Mr. Jones borrows \$500 and two years later returns \$540 to the lender. At what annual simple interest rate did he borrow the money?

17. A merchant sold a radio for \$60.00. If the radio cost him \$37.50 and his overhead is one-third of the cost, then what profit did he make?

18. Mr. Andrews is a salesman. He works at a commission rate of 15%. How much merchandise must he sell to collect a commission of \$750?

19. At a recent sale a department store "marked down" its prices by 40%. A man saved \$3.20 by buying a shirt during the sale. What did he pay for the shirt?

20. A farmer's field has four straight sides. The lengths of three sides are 1000 feet, 1500 feet, and 650 feet. The fourth side passes through a dense patch of brambles making it difficult to measure. The previous owner who had fenced the place before the brambles grew up told him that it took 4000 feet of fence to go around all four sides. How long is the fourth side?

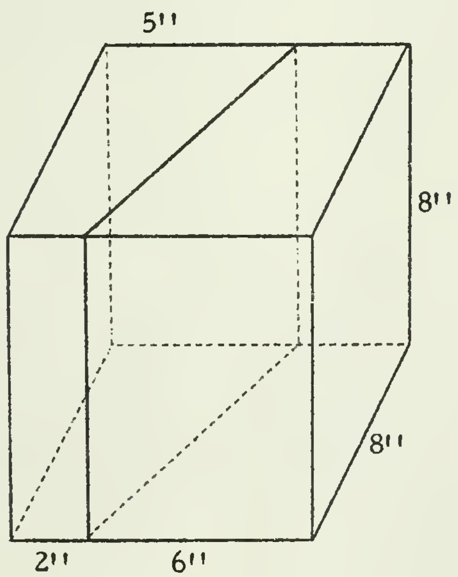
21. The reason that the farmer mentioned in Exercise 20 wants to know the length of the fourth side is that he is going to hire a heavy-duty rotary cultivator to come in and cultivate along the fence on that side, thereby getting rid of the brambles. The cultivator charges 5 cents for each foot cultivated. How much will it cost to have the fourth side cultivated?

22. A man borrows 1000 dollars at simple interest for five years. He pays a total of 250 dollars interest in that time. What yearly rate of interest was he paying?

23. A round table top made out of walnut wood has an area of 12 square feet. What is its radius? What is its diameter? What would be the radius if the table were made out of maple?

24. A square has a side of length 10 feet. This figure is sometimes called a ten foot square. How many square feet are there in this <u>ten foot square</u>?

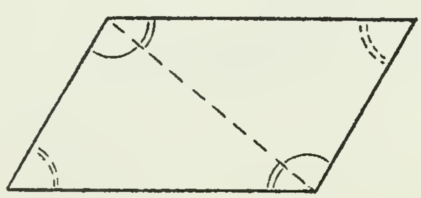
25. What is the largest amount of money that can be borrowed for two years at an annual interest rate of 5% if the total amount of interest cannot exceed $50 for the two years?



26. A block of wood is in the shape of a cube 8" on each edge. The cube is cut into two pieces by sawing "straight down" along a line drawn on top of the cube. Find the volume of each of the two pieces. (Neglect the loss due to sawdust.)

27. A right triangle has one angle of 42°. What is the size of its third angle?

28. What is the sum of the measures of the four angles in a parallelogram?



In Part A we have 8 pairs of equivalent equations. In each pair, when the student finds that the root of (b) satisfies (a), he may tend to search for some relationship between the two equations. Do not ask him to try to describe this relationship.

* * *

The four exercises in Part B are intended to familiarize the students with the basis underlying the equation transformation principle. The generalization in Exercise 2 is derivable from the one in Exercise 1 by virtue of the principle of subtraction. Exercise 1 tells you that if you add a number to a number there is just one sum, and Exercise 2 tells you that there is just one difference of a number from a number. In the system of directed numbers, these ideas are referred to by the statement:

Addition is a unique operation,

and Exercises 1 and 2 are expressions of the <u>uniqueness of addition principle</u>. [Students should learn the underlined phrase.]

The checking of these generalizations is a very simple matter when students select replacements which convert the antecedents of the conditionals into true statements. It is in these cases that the uniqueness notion becomes apparent. In those cases in which the antecedents are converted into false statements, the truth tables for conditionals [see T. C. 8V] tell you that the conditional is true (regardless of whether the consequent is true or false).

EXPLORATION EXERCISES

A. In each of the following exercises you are given a pair of equations. The first equation in each pair is not like any you have solved up to this point. Note that this equation has pronumerals in the expressions on both sides of the '='. On the other hand, the second equation in each pair is like the equations you have already solved. So, for each exercise, solve the second equation and then check to see if its roots satisfy the first equation.

1. (a) $8a - 6 = 2a$
   (b) $8a - 6 - 2a = 0$

2. (a) $4x + 2 = 3x$
   (b) $4x + 2 - 3x = 0$

3. (a) $11b - 8 = 7b$
   (b) $4b - 8 = 0$

4. (a) $12d + 7 = -2d$
   (b) $14d + 7 = 0$

5. (a) $5x - 6 = 3x + 8$
   (b) $2x - 6 = 8$

6. (a) $3y - 2 = 2y - 7$
   (b) $y - 2 = -7$

7. (a) $7t - 8 = 10 - 2t$
   (b) $9t - 8 = 10$

8. (a) $6s + 9 = -2 - 5s$
   (b) $11s + 9 = -2$

B. Check each of the following statements by substituting numerals for pronumerals.

1. For every $\square$, $\triangle$, and $\bigcirc$,

   if $\square = \triangle$, then $\square + \bigcirc = \triangle + \bigcirc$.

2. For every $\square$, $\triangle$, and $\bigcirc$,

   if $\square = \triangle$, then $\square - \bigcirc = \triangle - \bigcirc$.

(continued on next page)

'$\frac{1}{8 - 8}$' has no meaning] and cannot be called true. For each $x \neq 8$, $x + \frac{1}{x - 8} \neq 8 + \frac{1}{x - 8}$ since $\frac{1}{x - 8} = \frac{1}{x - 8}$. Hence, the solution set of the second equation is $\emptyset$.

10. (1) $\{x: xx = x\} \subseteq \{x: \frac{xx}{x} = \frac{x}{x}\}$ ..... false

(2) $\{x: \frac{xx}{x} = \frac{x}{x}\} \subseteq \{x: xx = x\}$ ..... true

11. Since the solution set of the second equation is the set of all directed numbers, we know at once that the solution set of the first equation is a subset of the solution set of the second. Also, since it is obvious upon inspection that 0 does not satisfy the first equation, its solution set does not include the solution set of the second equation. So,

$$\{x: 3xx - 2xxx + 5x - 7 = 8x - 9xx + 5\} \subseteq \{x: x + 1 = 1 + x\},$$

and:

$$\{x: x + 1 = 1 + x\} \not\subseteq \{x: 3xx - 2xxx + 5x - 7 = 8x - 9xx + 5\}.$$

[The notion that the empty set is a subset of every set may be difficult to explain, but you should try to explain it now with the thought in mind that students will return to this idea many times in later courses. Exercise 5 and earlier supplementary exercises have led students to accept the fact that there are solution sets which are empty. One's first reaction to the equation '$x = x + 1$' is that since it has no roots at all, it certainly does not have roots which satisfy '$2x = 160$', and that you would not say that $\{x: x = x + 1\} \subseteq \{x: 2x = 160\}$. But, when one asserts that a first set is not a subset of a second set, he is in fact asserting that there is at least one element of the first set which does not belong to the second set. In other words, one and only one of the following statements must be true:

(a) $\{x: x = x + 1\} \subseteq \{x: 2x = 16\}$,

(b) $\{x: x = x + 1\} \not\subseteq \{x: 2x = 16\}$.

It is easy to see that for (b) to be true, you would have to find a number which did not satisfy '$2x = 16$' and which did satisfy '$x = x + 1$'. No number fits this description because '$x = x + 1$' has no roots. So (b) is false, and, therefore, (a) is true.]

9. (1) $\{x: x = 8\} \subseteq \{x: x + \frac{1}{x - 8} = 8 + \frac{1}{x - 8}\}$ ..... false

(2) $\{x: x + \frac{1}{x - 8} = 8 + \frac{1}{x - 8}\} \subseteq \{x: x = 8\}$ ..... true

The solution set of the first equation is $\{8\}$ and that of the second equation is $\emptyset$. 8 is not a root of the second equation because:

$$8 + \frac{1}{8 - 8} = 8 + \frac{1}{8 - 8}$$

is a nonsense statement ['$8 + \frac{1}{8 - 8}$' is not a numeral since

(continued on T. C. 24E)

each exercise as we have done in the case of Exercises 1 and 3. Introduce the term 'equivalent equations' and explain it with the help of this notion of equality of solution sets.]

The equations in Exercise 5 are equivalent because the solution set of each is the empty set:

$$\{x: x + 1 = x + 2\} = \emptyset = \{x: x = x + 1\}.$$

Add these exercises to Part C.

7. $x = 8$
$2 \cdot |x| = |2x|$

8. $x = x + 1$
$2x = 160$

9. $x = 8$
$x + \frac{1}{x - 8} = 8 + \frac{1}{x - 8}$

10. $xx = x$
$\frac{xx}{x} = \frac{x}{x}$

11. $3xx - 2xxx + 5x - 7 = 8x - 9xx + 5$
$x + 1 = 1 + x$

Answers.

7. $\{x: x = 8\} \subseteq \{x: 2 \cdot |x| = |2x|\}$ but $\{x: 2 \cdot |x| = |2x|\} \not\subseteq \{x: x = 8\}$ because the solution set of '$2 \cdot |x| = |2x|$' consists of all directed numbers and the solution set of '$x = 8$' consists of just the number 8.

8. (1) $\{x: x = x + 1\} \subseteq \{x: 2x = 160\}$
(2) $\{x: 2x = 160\} \subseteq \{x: x = x + 1\}$

(1) is true because $\{x: x = x + 1\} = \emptyset$, and because [see below] there is no element in $\emptyset$ which is not contained in $\{x: 2x = 160\}$.
(2) is false because 80 is an element of $\{x: 2x = 160\}$ but, of course, not of $\{x: x = x + 1\}$.

(continued on T. C. 24D)

The truth of each of the instances of this new generalization is immediately decidable by means of the truth tables. The ''nonsensical'' instance referred to above is not an instance of this generalization.

* * *

The instructions in Part C require the students to compare the solution sets of the two equations in each exercise. Consider Exercise 1. The students must decide whether each of the following statements is true or false:

$$(1) \qquad \{x:\ xx = 16\} \subseteq \{x:\ x = 4\},$$

$$(2) \qquad \{x:\ x = 4\} \subseteq \{x:\ xx = 16\}.$$

In this case, (1) is false because -4 is an element of $\{x:\ xx = 16\}$ and is not an element of $\{x:\ x = 4\}$. (2) is true because each element of the singleton $\{x:\ x = 4\}$ belongs to $\{x:\ xx = 16\}$. [Another name for the solution set of '$xx = 16$' is '$\{x:\ x = 4 \text{ or } x = -4\}$'.] Now, consider Exercise 3. Here, both of the statements:

$$(1') \qquad \{y:\ yy = 100\} \subseteq \{y:\ |y| = 10\},$$

and: $$(2') \qquad \{y:\ |y| = 10\} \subseteq \{y:\ yy = 100\}$$

are true. In such a case we can assert that

$$\{y:\ yy = 100\} = \{y:\ |y| = 10\}$$

[for each set α and each set β, $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$ if and only if $\alpha = \beta$], which is another way of saying the same thing as:

'$yy = 100$' and '$|y| = 10$' are equivalent equations,

since equivalent equations are equations which have the same roots. [Since we shall make frequent use of the notion of equality of solution sets throughout this unit, we suggest that you have the students discuss

(continued on T. C. 24C)

Part B (continued)

Exercises 3 and 4 are expressions of the uniqueness of multiplication principle.

Ask students if they can find an instance of the generalization in Exercise 3 which has a false antecedent and a true consequent [Example: if $7 = 9$ then $7 \times 0 = 9 \times 0$].

In checking instances of Exercise 4 be sure to discuss the 0-restriction. If the 0-restriction is violated, one might obtain as an instance the following conditional:

(*) if $0 \neq 0$ and $7 = 7$ then $7 \div 0 = 7 \div 0$.

The consequent of (*) is something which is, grammatically, a sentence but which contains two occurrences of '$7 \div 0$', a mark which has no meaning. Since '$7 \div 0 = 7 \div 0$' is neither true nor false [but just nonsense], our truth tables for conditionals [T. C. 8V] do not tell us whether (*) is true or false. [We could decide arbitrarily that '$7 \div 0 = 7 \div 0$' is true, or we could decide that it is false. In either case, since the antecedent of (*) is false, (*) is true.] But in order to avoid a situation such as this, we shall rewrite the generalization in such a way that 0 is eliminated from the domain of '$\bigcirc$':

For every $\square$, $\triangle$, and $\bigcirc \neq 0$,

if $\square = \triangle$ then $\square \div \bigcirc = \triangle \div \bigcirc$.

[Students should make this change in their texts.]

(continued on T. C. 24B)

3. For every $\square$, $\triangle$, and $\bigcirc$,

   if $\square = \triangle$, then $\square \times \bigcirc = \triangle \times \bigcirc$.

4. For every $\square$, $\triangle$, and $\bigcirc$,

   if $\bigcirc \neq 0$ and $\square = \triangle$,

   then $\square \div \bigcirc = \triangle \div \bigcirc$.

C. Each of the following exercises contains a pair of equations. In each exercise tell (1) whether all the roots of the first equation are also roots of the second equation, and (2) whether all the roots of the second equation are also roots of the first equation.

1. $xx = 16$
   $x = 4$

2. $3xx = 75$
   $x = -5$

3. $yy = 100$
   $|y| = 10$

4. $9aa = 36$
   $aa = 4$

5. $x + 1 = x + 2$
   $x = x + 1$

6. $xx = x$
   $x = 1$

3.05 <u>Solving more difficult equations</u>.--As you have observed, an equation contains two algebraic expressions separated by an equal-sign. Each of these two algebraic expressions is called a <u>member of the equation</u>. Almost all of the equations that you have solved so far have had the following property: only one of the two members contains pronumerals. Equations of that type usually are easy to solve. You have probably invented your own methods for solving them by now.

For every m,

$$\boxed{3m - 2} = 8 - 2m$$

if and only if

$$\boxed{3m - 2} + 2m = 8 - 2m + 2m \;.$$

$$5m - 2 = 8$$

The root is 2.

* * *

You have no doubt noted that instead of saying such things as 'add 6 to both sides of the equation', we say, 'write a '+6' on both sides of the equation'. The avoidance of the first kind of expression is in keeping with the notion that the only things which can be added are numbers. Since a side of an equation is an expression, rather than a number, we do not want to talk about adding a number to a side of an equation. In your own exposition, you should avoid this usage and adopt ours. However, you will find that your students do not hesitate to say 'add 6 to both sides'. Although you may call their attention to this technical error, it is foolhardy to try to get them to be precise. After all, they are really thinking mathematics out loud and for that reason are justified in saying 'add' If you use the right mode of speech consistently, and gently correct students just every so often, no harm will come of their saying 'add ... to both sides'.

uniqueness of addition principle is the basic principle used in <u>transforming</u> (1) into (2), we call this principle <u>the addition transformation principle</u>.

The addition transformation principle is applied by using the following <u>pattern</u>.

For every ... ,

$$\square = \bigcirc$$

if and only if

$$\square + \triangle = \bigcirc + \triangle \,,$$

where the frames are replaced by numerals or by expressions which contain occurrences of the pronumeral [or pronumerals] indicated by the quantifier. As soon as the frames are filled and the ' ... ' replaced by the proper pronumeral [or pronumerals], the above pattern is converted into a <u>case</u> of the addition transformation principle. The two components of the quantified biconditional are equivalent equations, or equations which have the same solution set. Students should copy this pattern into their texts on the page facing page 3-27, and label it 'pattern used in applying the addition transformation principle'.
In assigning Parts A and B on page 3-29, tell the students to ignore the instructions and the Samples. Instead, they should complete the pattern for each exercise, and then find the roots. For example, here is how they might handle Exercise 1 of Part A on their homework papers.

(continued on T. C. 25I)

For every x, y, and z,

$$x = y$$

if and only if

$$x + z = y + z.$$

Instances of this generalization which are obtained by replacing all occurrences of the three pronumerals by numerals are statements called <u>biconditionals</u>. Biconditionals are true when their two components are both true or both false. The particular <u>case</u> of the principle of uniqueness of addition which tells us that (1) and (2) are equivalent equations is:

For every $\square$,

$$5 \square - 3 = 7 - 4 \square$$

if and only if

$$[5 \square - 3] + 4 \square = [7 - 4 \square] + 4 \square .$$

Clearly, each instance of this generalization is an instance of the principle of uniqueness of addition. Since we regard this principle as true, we regard as true all of its instances. So, all of the instances of the case in question are true; this means that the equations

$$‘5 \square - 3 = 7 - 4 \square’$$

and

$$‘[5 \square - 3] + 4 \square = [7 - 4 \square] + 4 \square’$$

can be converted into pairs of statements which are either both true or both false. Hence, these equations are equivalent. Because the

(continued on T. C. 25H)

its members into names for the same number, that root of (2) will convert the members of (3) into names for that very same number. In like manner, you can argue that each root of (3) is a root of (2). Thus, the solution set of (2) is the solution set of (3); that is, (2) and (3) are equivalent equations. Students should be told that (2) and (3) are equivalent equations by virtue of the principle of equivalent algebraic expressions. But, we still need to answer the question concerning the fact that (1) and (3) are equivalent equations. Since (1) and (2) have equal solution sets, and since (2) and (3) have equal solution sets, it follows from the principle of equality [see Unit 2, T. C. 59B] that (1) and (3) have equal solution sets.

As we have seen, the fact that (1) and (2) have equal solution sets follows from the uniqueness of addition principle and the uniqueness of multiplication principle. The more important of these principles in this case, and the one which is suggestive of the deriving step, is the uniqueness of addition principle. Therefore, students should justify the assertion that (1) and (2) are equivalent equations by citing the uniqueness of addition principle. It is now time to state this principle. Clearly, there are two "parts" to it:

For every x, y, and z,
if $x = y$ then $x + z = y + z$,

and:

For every x, y, and z,
if $x + z = y + z$ then $x = y$.

Students should be told that a more compact way of stating this principle is:

(continued on T. C. 25G)

$$10 + 7x = 9x + 6$$
$$4 - y = 3y + 7$$

you can construct several more (about 10) equations like these and assign them for homework.

When the students return to class the next day, check their homework immediately to make sure that the technique has been reasonably mastered. Then, with the very active help of the students, repeat the arguments you developed the preceding day, using one of their homework exercises as an example. Then, consider the first of the two problems mentioned above. [The second will be considered in the Commentary for page 2-31.]

How do we know that the solution set of (1) is the same as the solution set of (3)?

(1) $5\square - 3 = 7 - 4\square$

(2) $[5\square - 3] + 4\square = [7 - 4\square] + 4\square$

(3) $9\square - 3 = 7$

The left members of (2) and (3) are equivalent algebraic expressions. This means that, for every $\square$, $[5\square - 3] + 4\square = 9\square - 3$. Or, for each replacement of the '$\square$' in '$[5\square - 3] + 4\square$' and in '$9\square - 3$', you get names for the same number. Similarly, for each replacement of the '$\square$' in '$[7 - 4\square] + 4\square$' and in '7', you get names for the same number. Hence, since a root of (2) converts

(continued on T. C. 25F)

exceeds the number obtained from (1) by 4 times this root of (1). Since, by the uniqueness of multiplication principle, there is only one number which is 4 times this root of (1), and since, by the uniqueness of addition principle, there is only one number which exceeds the number obtained from (1) by 4 times this root of (1), a root of (1) satisfies (2). Similarly, a root of (2) satisfies (1) [uniqueness of multiplication (-4 times a root of (2)), and uniqueness of addition (add -4 times a root of (2))]. But, simplifying (2) yields:

$$(3) \qquad 9 \boxed{} - 3 = 7,$$

which is satisfied by $\frac{10}{9}$. [Repeat the argument for the case of writing '$-5 \boxed{}$' on both sides.]

There are still two problems to be considered. First, how do we know that the solution set of (3) is the solution set of (1)? And, secondly, how do we know that the solution set of (1) is the singleton $\{\frac{10}{9}\}$?

Before pursuing these questions with the students, you should give them several equations of the type exemplified by (1) in order to familiarize them with the technique. One or two of these should be worked by you at the blackboard using suggestions drawn from the students; the rest should be worked by students at their seats and checked by you as you walk among the students. Here are several equations to use for this purpose.

$$3\triangle + 7 = 9 - 8\triangle$$
$$5x - 4 = 7x + 3$$
$$2k + 12 = 22 - 3k$$

(continued on T. C. 25E)

Therefore, the s[illegible]
This means, if [illegible] roots of (1) [illegible]
by finding [illegible]
case of the [illegible] (2) [illegible]
simplified to giv[illegible]

$$5 \; [illegible]$$

an equation [illegible]

[illegible]

simplified [illegible]

lines. W[illegible]

writ[illegible]

[illegible]

For each [illegible]

[illegible] 256, [illegible]

Therefore, the solution set of (1) is equal to the solution set of (2). This means, of course, that we can find all the roots of (1) just by finding all the roots of (2). We examine (2) in detail. As in the case of the equations on page 3-13, the members of (2) can be simplified to give:

$$5 \square + 3 = 13 - 4 \square ,$$

an equation which is no easier to solve than (1). But, in deriving (2) from (1), we have used a method which produces an equation equivalent to (1). By a more clever application of this method it is possible to derive an equation from (1) which is equivalent to (1) and which <u>is</u> easier to solve. We see that the difficulty in solving (1) is that a pronumeral occurs in both members. Is there something we can write on both sides of (1) which will yield an equation which when simplified will be easier to solve than (1)? There are two possibilities. We could write '$+ 4 \square$' on each side, or we could write '$- 5 \square$' on each side. In either case, the simplification of the resulting equation would result in an equation one of whose members is an expression in which a pronumeral does not occur. We try '$+ 4 \square$'.

$$(1) \qquad 5 \square - 3 = 7 - 4 \square$$

$$(2) \qquad [5 \square - 3] + 4 \square = [7 - 4 \square] + 4 \square$$

Once again, we show that the solution set of (1) is the same as the solution set of (2). A root of (1) whose name is written in each occurrence of the '$\square$' in (2) converts (2) into a true statement.

For each converted member of (2) is a name for a number which

(continued on T. C. 25D)

(1) $5 \boxed{} - 3 = 7 - 4 \boxed{}$

(2) $[5 \boxed{} - 3] + 6 = [7 - 4 \boxed{}] + 6.$

Is a number which is a root of (1) also a root of (2)? The answer to this question is 'yes', and the reason for it is important as well as easy to see. When a name for a root is written in each occurrence of the '$\boxed{}$' in (1), (1) is converted into a true statement. That is, '$5 \boxed{} - 3$' and '$7 - 4 \boxed{}$' are converted into <u>different names for the same number</u>. When a name for this root of (1) is written in each occurrence of the '$\boxed{}$' in (2), <u>both</u> members of (2) are converted into names for the number which is 6 more than the number obtained in the case of (1). By the uniqueness of addition principle, the statement obtained from (2) must be true. Hence, each root of (1) is a root of (2), or, in other words, the solution set of (1) is a <u>subset</u> of the solution set of (2).

We still have not shown that the solution sets of (1) and (2) are equal. So now we ask if each root of (2) is a root of (1). Once again, writing a name for a root of (2) in each occurrence of the '$\boxed{}$' in (2) converts the members of (2) into different names for the same number. Writing names for a root of (2) in each occurrence of the '$\boxed{}$' in (1) converts both members of (1) into names for the number which is 6 less than the number obtained in the case of (2). By the uniqueness of addition principle ["adding -6"], the statement thus obtained from (1) is true. Hence, the solution set of (2) is a subset of the solution set of (1).

(continued on T. C. 25C)

[This Commentary should be used for pages 3-25 through the top half of 3-30.]

* * *

In classes at University High School and in demonstration classes in Project schools, we have developed techniques for teaching the principles of equation transformation which are superior to the techniques indicated in the students' text. Naturally, the students' text will embody these new techniques in our next revision, but until that time, we suggest that you ignore the text almost completely, and use instead the procedure described below. [It is impossible to include in the text a development that can match in effectiveness one carried out by a teacher who is free to point at expressions on the board, and who can draw links and arrows with abandon.]

After playing with the equation:

$$(1) \quad 5 \boxed{} - 3 = 7 - 4 \boxed{}$$

for a while in order to discover that the roots of (1) are not as easy to find as in the case of earlier equations, the teacher announces that $\frac{10}{9}$ is the root of (1). He tells the class that he discovered this fact by deriving from (1) another equation which was easy to solve, and whose solution set was the same as that of (1). The trick is: how to derive this simpler equation?

Consider both the given equation and (2), the one obtained by writing a '+6' on the right end of each member of (1);

(continued on T. C. 25B)

Here is an example of an equation with pronumerals in both members:

$$5 \square - 3 = 7 - 4 \square .$$

Perhaps you can solve this equation already. (Try it.) Now you will learn a method which makes an easy task of solving an equation such as the one above.

EQUIVALENT EQUATIONS

Suppose that you have found a root for the equation:

$$5 \square - 3 = 7 - 4 \square .$$

If you put a numeral for the root in each '$\square$' of the equation, you will obtain a true statement. Here is another way of saying what a root is. If you write a numeral for the root in the '$\square$' of '$5 \square - 3$' and in the '$\square$' of '$7 - 4 \square$', you will obtain two expressions for the same number.

Now let us change the two expressions

'$5 \square - 3$' and '$7 - 4 \square$'

by writing '$+ 4 \square$' on the right of each of them. We obtain

'$5 \square - 3 + 4 \square$' and '$7 - 4 \square + 4 \square$'.

You can see that if you write a numeral for a root of the original equation in each '$\square$' of the two new expressions above, you will also get two expressions for the same number. In other words, the following two equations have the same roots:

(1) $5 \square - 3 = 7 - 4 \square$

(2) $5 \square - 3 + 4 \square = 7 - 4 \square + 4 \square$.

Now, let us examine equation (2) more closely. An expression which is equivalent to but simpler than its left member is '$9 \square - 3$'; an expression which is equivalent to but simpler than its right member is '7'. Therefore,

we can solve equation (2) by solving:

$$(3) \qquad 9\,\square - 3 = 7$$

because equation (2) and equation (3) have the same roots. A root of equation (3) is $\frac{10}{9}$. Let's check to see that $\frac{10}{9}$ is also a root of equation (1).

$5(1\frac{1}{9}) - 3$	$7 - 4(1\frac{1}{9})$
$= 5\frac{5}{9} - 3$	$= 7 - 4\frac{4}{9}$
$= 2\frac{5}{9}$	$= 2\frac{5}{9}$

So, $\frac{10}{9}$ is a root of the equation we wanted to solve at the outset.

Let us summarize what we have done. We started with the equation:

$$5\,\square - 3 = 7 - 4\,\square$$

and derived from this equation the new equation:

$$9\,\square - 3 = 7.$$

The new equation is of the type we have solved before; it has a pronumeral in only one of its members. The new equation and the given equation have the same root. In deriving the new equation from the given equation we used the following two principles:

> For every $\square$,
>
> if $5\,\square - 3 = 7 - 4\,\square$,
>
> then $5\,\square - 3 + 4\,\square = 7 - 4\,\square + 4\,\square$
>
> or, simply,
>
> $9\,\square - 3 = 7.$

> For every $\square$,
>
> if $9\,\square - 3 = 7$,
>
> then $9\,\square - 3 + (-4\,\square) = 7 + (-4\,\square)$
>
> or, simply,
>
> $5\,\square - 3 = 7 - 4\,\square.$

The principle stated in the first box above tells you that the roots of '5 $\square$ - 3 = 7 - 4 $\square$ ' are roots of '9 $\square$ - 3 = 7'; the principle stated in the second box tells you that the roots of '9 $\square$ - 3 = 7' are roots of '5 $\square$ - 3 = 7 - 4 $\square$ '. Equations like '5 $\square$ - 3 = 7 - 4 $\square$ ' and '9 $\square$ - 3 = 7' each of which is satisfied by the same numbers (in this case the single number $\frac{10}{9}$) are called equivalent equations. So, if you are given an equation which has pronumerals in both members, you can solve this equation simply by deriving an equation which is equivalent to the given equation but which contains pronumerals in only one member and then solving this derived equation. Deriving an equation which is equivalent to a given equation is called transforming the given equation. The principles which tell you how to derive an equivalent equation from a given equation are known as equation transformation principles. Here is the first transformation principle. We shall call it the principle of transforming an equation by addition or briefly the addition principle:

> For every X, Y, and Z,
> X = Y
> if and only if
> X + Z = Y + Z.

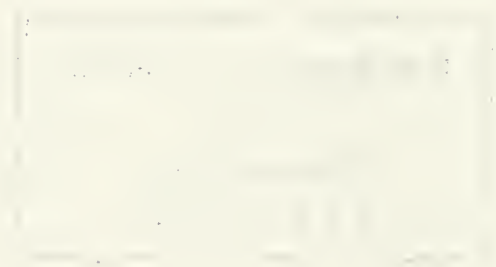
(We shall state another transformation principle later.) The phrase 'if and only if' tells you that 'X = Y' is satisfied by the same numbers which satisfy 'X + Z = Y + Z' and that 'X + Z = Y + Z' is satisfied by the same numbers which satisfy 'X = Y'. That is, 'X = Y' and 'X + Z = Y + Z' are equivalent equations according to this principle.

Let us see how this principle helps us solve equations.

Example 1. Solve: $18a - 5 = 9 + 4a$

Solution. We want to transform this equation to an equivalent equation which contains a pronumeral in only one member. To see how to use the addition

(continued on next page)



principle we look at, say, the right member of the equation. If we could change '9 + 4a' to '9 + 4a - 4a', we could then simplify the right member to '9'. Using the addition principle and replacing 'X' by '18a - 5', 'Y' by '9 + 4a', and 'Z' by '-4a', we obtain the following:

$$18a - 5 = 9 + 4a$$

if and only if

$$18a - 5 + (-4a) = 9 + 4a + (-4a)$$

which, according to the transformation principle, leads to true statements for each replacement of 'a'. The derived equation:

$$18a - 5 + (-4a) = 9 + 4a + (-4a)$$

has members which can be simplified to give:

$$14a - 5 = 9.$$

This equation has a pronumeral in only one member and according to the transformation principle, it is equivalent to the given equation. So, we can solve the equation:

$$14a - 5 = 9$$

and be assured that its roots are the roots of the given equation. You can solve '14a - 5 = 9' by the methods you discovered earlier in the Unit. This equation has the single root 1. We know from the transformation principle (if we have not made an error) that 1 is the only root of the given equation. We can check partially against errors by substituting '1' for 'a' in the given equation:

$18(1) - 5$	$9 + 4(1)$
$= 18 - 5$	$= 9 + 4$
$= 13$	$= 13$

EXERCISES

A. Solve these equations by first transforming them to equations which have pronumerals in one member only. Tell how you are using the addition principle. That is, tell what expressions replace 'X', 'Y', and 'Z' in the statement of the principle on page 3-27.

Sample. $6a - 3 = 14a + 20$

Solution. Look at the statement of the addition principle. Replace

'X' by '6a - 3'
'Y' by '14a + 20'
'Z' by '-6a'

and you obtain:

$$6a - 3 = 14a + 20$$

if and only if

$$6a - 3 + (-6a) = 14a + 20 + (-6a).$$

Simplify this equation to:

$$-3 = 8a + 20.$$

The root of this equation is $-\frac{23}{8}$. Then $-\frac{23}{8}$ is also a root of the original equation but you should check it by substitution to help you catch errors you may have made.

1. $3m - 2 = 8 - 2m$
2. $7t - 5 = 15 + 3t$
3. $4 + 5s = 3s$
4. $8 - k = 5k - 10$
5. $3 + 2x = 7 + 9x$
6. $5y - 3 = 8 + 2y$
7. $4m - 3 = 3 - 4m$
8. $3k - 7 = 4\frac{1}{2} - 12k$

B. The equations in this part contain pronumerals in one member only. You could solve these equations before you learned the principle of transforming equations by addition. You can also solve these simple equations by using the addition principle to derive equivalent but even simpler equations. For a thorough understanding of the addition principle you should practice using it on simple equations.

Each of the generalizations in the Exploration Exercises has an instance which is a conditional with a nonsensical consequent. As indicated on T. C. 24 it will be best to avoid this situation by restricting the quantifiers. Thus, each exercise should be rewritten. For example, rewrite Exercise 1 as:

$$\text{For every } m \neq 0,\ \frac{5}{m} \times m = 5,$$

and Exercise 13 as:

$$\text{For every } t \neq 0 \text{ and } \neq 3,\ \ldots\ .$$

Students should recognize quickly that each of these generalizations [except 8] is based primarily on the fact that multiplication and division are inverse operations. The distributive, associative, commutative principles also come into play in some of the exercises.

Solve the following equations using the addition principle and, as in Part A, tell what expressions replace 'X', 'Y', and 'Z'.

Sample. $a + 3 = 8$

Solution. Use '$a + 3$' for 'X', '8' for 'Y', and '-3' for 'Z'. Then, according to the principle, we derive from:

$$a + 3 = 8$$

the equivalent equation:

$$a + 3 + (-3) = 8 + (-3)$$

or, simply,

$$a = 5.$$

Clearly, the equation '$a = 5$' has but one root, 5; this number also satisfies the given equation '$a + 3 = 8$'.

1. $c + 9 = 24$
2. $d - 4 = 12$
3. $5 + r = 16$
4. $7 = t - 3$

EXPLORATION EXERCISES

The following statements are true. (Except one, find it!) You should make enough replacements in each statement to convince yourself that it is true. Study the general "pattern" until you can write more true statements like these.

1. For every m, if $m \neq 0$, then $\frac{5}{m} \times m = 5$.

2. For every k, if $k \neq 0$, then $(\frac{7 - k}{k})(k) = 7 - k$.

3. For every t, if $t \neq 0$, then $(\frac{5t - 6}{t})(t) = 5t - 6$.

4. For every s, if $s \neq 5$, then $(\frac{s + 6}{s - 5})(s - 5) = s + 6$.

5. For every r, if $r \neq 0$, then $(\frac{3}{r} + \frac{2}{r})(r) = 3 + 2$.

6. For every x, if $x \neq 0$, then $(\frac{4}{x} + 9)(x) = 4 + 9x$.

(continued on next page)

$$(*) \qquad (x - 7)(x - 3) \times \frac{1}{x - 3} = 0 \times \frac{1}{x - 3} \qquad \text{[m. t. p.]}$$

$$x - 7 = 0 \qquad \text{[p. e. a. e.]}$$

$$x - 7 + 7 = 0 + 7 \qquad \text{[a. t. p.]}$$

$$x = 7 \qquad \text{[p. e. a. e.]}$$

The root of (1) is 7.

Note that (*) is a correct application of the m. t. p. because the expression '$\frac{1}{x - 3}$' makes sense if 3 is excluded from the domain of 'x', and this exclusion took place in using the m. t. p. to derive (2) from (1). [In a sense, you could say that 3 was excluded from the domain of 'x' when the equation (1) was first constructed. This point is illustrated by the following application of the addition transformation principle:

$$(3) \quad 2x = 10$$

$$(4) \quad 2x + \frac{1}{x - 5} = 10 + \frac{1}{x - 5}.$$

In using the expression '$\frac{1}{x - 5}$' one must assume that 5 has been excluded from the domain of 'x'. Under this assumption, we know from the a. t. p. that (3) and (4) are equivalent with respect to the set of all directed numbers other than 5. We can express this fact readily by using the solution set notation:

$$\{x: x \neq 5 \text{ and } 2x = 10\} = \{x: x \neq 5 \text{ and } 2x + \frac{1}{x - 5} = 10 + \frac{1}{x - 5}\}.$$

Since each of the members of the foregoing extended equation names the same set, and since $\{x: x = \frac{10}{9}\} = \{\frac{10}{9}\}$, then the root of '$5x - 3 = 7 - 4x$' is $\frac{10}{9}$.

* * *

Careful use of the multiplication transformation principle removes some of the traditional "mysteries" about "extraneous roots" and "lost roots". For example, the equation:

$$(1) \quad x + \frac{5x}{x - 3} = 3\left[4 + \frac{5}{x - 3}\right]$$

and the equation:

$$(2) \quad x(x - 3) + 5x = 12(x - 3) + 15$$

are not equivalent, even though (2) is derivable from (1) by an application of the m. t. p. However, one can say that <u>they are equivalent with respect to the set of all directed numbers excluding 3</u>. Or, one can say that

$$\{x: x \neq 3 \text{ and } x + \frac{5x}{x - 3} = 3\left[4 + \frac{5}{x - 3}\right]\}$$

$$= \{x: x \neq 3 \text{ and } x(x - 3) + 5x = 12(x - 3) + 15\}.$$

To complete the solution of (1):

$$\begin{array}{rcll} x(x - 3) + 5x & = & 12(x - 3) + 15, \quad [x \neq 3] & \\ x^2 + 2x & = & 12x - 21 & [\text{p. e. a. e.}] \\ x^2 + 2x - [12x - 21] & = & 12x - 21 - [12x - 21] & [\text{a. t. p.}] \\ x^2 - 10x + 21 & = & 0 & [\text{p. e. a. e.}] \\ (x - 7)(x - 3) & = & 0 & [\text{p. e. a. e.}] \end{array}$$

(continued on T. C. 31G)

For every ..... [$\neq 0$],

$$\square = \bigcirc$$

if and only if

$$\square \times \triangle = \square \times \triangle .$$

If the expression to be written in each '$\triangle$' does not contain a pronumeral, then no zero-restriction is necessary.

Examples 2 [page 3-33] and 3 [page 3-35] should be handled as they are in the text, except that students should use the frames (as above) instead of 'X', 'Y', and 'Z'. Note the need for the zero-restriction in Example 3.

* * *

We can now answer the second of the two questions asked on T. C. 25D:

How do we know that $\frac{10}{9}$ is the <u>only</u> root of '$5x - 3 = 7 - 4x$'?

$$
\begin{array}{lll}
 & \{x:\ 5x - 3 = 7 - 4x\} & \\
= & \{x:\ 5x - 3 + 4x = 7 - 4x + 4x\} & [\text{a. t. p.}] \\
= & \{x:\ 9x - 3 = 7\} & [\text{p. e. a. e.}] \\
= & \{x:\ 9x - 3 + 3 = 7 + 3\} & [\text{a. t. p.}] \\
= & \{x:\ 9x = 10\} & [\text{p. e. a. e.}] \\
= & \{x:\ 9x \times \frac{1}{9} = 10 \times \frac{1}{9}\} & [\text{m. t. p.}] \\
= & \{x:\ x = \frac{10}{9}\} & [\text{p. e. a. e.}]
\end{array}
$$

(continued on T. C. 31F)

or:

$$\{a:\ a \neq 0 \text{ and } 2a = 6\} = \{a:\ a \neq 0 \text{ and } 2aa = 6a\}.$$

In the text example [pages 3-31 and 3-32], the equations '$\frac{a - 2}{a} = 5$', '$\left[\frac{a - 2}{a}\right]a = [5]a$', and '$a - 2 = 5a$' are equivalent with respect to the set of non-zero directed numbers, or

$$\{a:\ a \neq 0 \text{ and } \frac{a - 2}{a} = 5\} = \{a:\ a \neq 0 \text{ and } \left[\frac{a - 2}{a}\right]a = [5]a\}$$

$$= \{a:\ a \neq 0 \text{ and } a - 2 = 5a\}$$

In practice, one simply notes the "0-restrictions":

$\frac{a - 2}{a} = 5$	
$\left[\frac{a - 2}{a}\right]a = [5]a,\ a \neq 0$	[multiplication transformation principle]
$a - 2 = 5a$	[principle of equivalent algebraic expressions]
$a - 2 + (-a) = 5a + (-a)$	[addition transformation principle]
$-2 = 4a$	[principle of equivalent algebraic expressions]
$-2 \times \frac{1}{4} = 4a \times \frac{1}{4}$	[multiplication transformation principle]
$-\frac{1}{2} = a$	[principle of equivalent algebraic expressions]

The root is $-\frac{1}{2}$.

The pattern to be followed in applying the multiplication transformation principle is this:

(continued on T. C. 31E)

nat[illegible]al for 0, we can eliminate [illegible]

the generalization:

(ii) For every x and [illegible]

[illegible]

If [illegible] techniques of [illegible] [illegible] something [illegible]:

(i') [illegible]

[illegible]

and condition (i) [illegible]

(iii) [illegible]

[illegible]

we have a principle [illegible] equivalent [illegible]. (iii) is called the [illegible] principle. [S[illegible] should replace [illegible] box [illegible] (iii).]

A case of [illegible]

[illegible]

[illegible]

The equations [illegible] the domain [illegible] zero directed [illegible] numbers. [illegible] must include [illegible] [illegible] [illegible] [illegible]

[illegible]

[illegible] the s[illegible] zero directed [illegible]

[illegible] on [illegible]

numeral for 0, we can eliminate the false instances by modifying the generalization:

> (ii) For every x and y, and every $z \neq 0$,
> if $xz = yz$ then $x = y$.

If we modify the uniqueness of multiplication principle, (i), by similarly restricting its quantifier:

> (i′) For every x and y, and every $z \neq 0$,
> if $x = y$ then $xz = yz$,

and combine (i′) with (ii):

> (iii) For every x and y, and every $z \neq 0$,
> $x = y$ if and only if $xz = yz$,

we have a principle which can be used in deriving equivalent equations. (iii) is called the <u>multiplication transformation principle</u>. [Students should replace the boxed statement on page 3-33 with (iii).]

A case of (iii) is:

> For every $a \neq 0$,
> $2a = 6$ if and only if $2aa = 6a$.

The equations '$2a = 6$' and '$2aa = 6a$' are equivalent with respect to the domain of 'a'; in this case, the domain of 'a' is the set of all non-zero directed numbers, and <u>not</u> the set of all directed numbers. Clearly, the concept of equivalent equations must include the notion of a domain of the pronumerals used in the equation. Similarly, the concept of solution set includes the notion of domain. In the case at hand, we can say:

> '$2a = 6$' and '$2aa = 6a$' are equivalent with respect to the set of non-zero directed numbers,

(continued on T. C. 31D)

[This Commentary covers pages 3-31 through 3-36.]

As in the case of the development of the addition transformation principle, you should, largely, ignore the text and use the following development [in a somewhat expanded form].

The fact that there is an <u>addition</u> transformation principle suggests that there might be a multiplication transformation principle. Such a principle should enable us to transform an equation into one which is equivalent to it and is, perhaps, simpler to solve. For example, by writing '$\times$ 8' on both sides, we can transform the equation:

$$(1) \quad \frac{y}{8} = 5$$

into:

$$(2) \quad \left[\frac{y}{8}\right] \times 8 = [5] \times 8,$$

which simplifies to:

$$(3) \quad y = 40.$$

Clearly, (1), (2), and (3) have the same solution set; (1) is equivalent to (2) by virtue of the uniqueness of multiplication principle ["$\times$ 8" <u>and</u> "$\times \frac{1}{8}$"], (2) is equivalent to (3) by virtue of the principle of equivalent algebraic expressions, and (1) is equivalent to (3) by the principle of equality [of solution sets].

One might be tempted to state the uniqueness of multiplication principle as we did the corresponding addition principle:

For every x, y, and z,
$x = y$ if and only if $xz = yz$.

(continued on T. C. 31B)

7. For every y, if $y \neq 2$, then $(\frac{3}{y - 2} - 7)(y - 2) = 3 - 7(y - 2)$.

8. For every Z, if $Z \neq -3$, then $(4 + \frac{5}{Z + 3})(Z + 3) = 4(Z + 3) + 5Z$.

9. For every b, if $b \neq 0$, then $(5b) \div (b) = 5$.

10. For every c, if $c \neq 0$, then $(3c) \div (c) = 3$.

11. For every d, if $d \neq 2$, then $[4(d - 2)] \div (d - 2) = 4$.

12. For every p, if $p \neq -1$, then $[9(p + 1)] \div (p + 1) = 9$.

13. For every t, if $t \neq 0$ and $t \neq 3$,

    then $[\frac{1}{t} + \frac{1}{t - 3}][t(t - 3)] = t - 3 + t$.

14. For every s, if $s \neq 0$ and $s \neq -5$,

    then $(\frac{5}{s} - \frac{6}{s + 5})[s(s + 5)] = 5(s + 5) - 6s$.

15. For every m, if $m \neq 3$ and $m \neq -3$,

    then $(\frac{4}{m + 3} + \frac{7}{m - 3})[(m - 3)(m + 3)] = 4(m - 3) + 7(m + 3)$.

A SECOND TRANSFORMATION PRINCIPLE

Consider the equation:

$$\frac{a - 2}{a} = 5 .$$

Undoubtedly, if you worked long enough, you could invent a method to solve this equation. However, there is another transformation principle which can be used to derive an equivalent second equation which is easier to solve than the given equation.

The equation:

$$\frac{a - 2}{a} = 5$$

may seem difficult to solve because one of its members contains an algebraic fraction with pronumerals in its numerator and denominator. A simpler equation which is equivalent to it is:

$$a - 2 = 5a.$$

Mechanically, you obtain the simpler equation as follows:

Look at:

$$\frac{a - 2}{a} = 5 .$$

Write '(a)' (or '× a') on the right of each of its members:

$$\frac{a - 2}{a}(a) = 5(a) .$$

Simplify each member to obtain:

$$a - 2 = 5a.$$

You can solve this last equation by using the addition principle.

Do you see that the equations:

$$\frac{a - 2}{a}(a) = 5(a)$$

and

$$a - 2 = 5a$$

are equivalent? That is, that both equations are satisfied by the same number?

The right members of these two equations are equivalent expressions. But, the left members are not equivalent expressions because if 'a' in '$\frac{a - 2}{a}(a)$' is replaced by a numeral for 0, the result is meaningless. However, for all replacements of 'a' other than by a numeral for 0, '$\frac{a - 2}{a}(a)$' and 'a - 2' become names for the same number. But, since 0 is not a root of 'a - 2 = 5a', the two equations are equivalent, even though the left members are not equivalent expressions.

We say that from the equation:

$$\frac{a - 2}{a} = 5$$

we derive the equivalent equation:

$$a - 2 = 5a.$$

Check to see that these equations are equivalent by solving the derived equation and determining if its root is a root of the original equation.

Here is the principle which tells you that the original equation and the derived equation are equivalent:

For every X, Y, and Z, if $Z \neq 0$, then $X = Y$ if and only if $XZ = YZ$.

We call this second transformation principle <u>the principle of transforming an equation by multiplication</u>, or briefly, <u>the multiplication principle</u>. We shall apply the multiplication principle in several examples.

<u>Example 2</u>. Solve the equation:

$$\frac{a}{2} + 2 + \frac{a}{6} = 7 + \frac{a}{3}.$$

<u>Solution</u>. If we examine this equation, we see that the algebraic fractions in the equation make it at first seem difficult to solve. We shall transform it into another equivalent equation which does not contain fractions. We will be able to solve the derived equation easily. We know that for every a,

$$\begin{aligned} &\left(\frac{a}{2} + 2 + \frac{a}{6}\right)(6) \\ &= 6\left(\frac{a}{2}\right) + 6(2) + 6\left(\frac{a}{6}\right) \\ &= 3a + 12 + a. \end{aligned}$$

We also know that for every a,

$$\begin{aligned} &\left(7 + \frac{a}{3}\right)(6) \\ &= 6(7) + 6\left(\frac{a}{3}\right) \\ &= 42 + 2a. \end{aligned}$$

We now have a way of transforming the original equation to give us a simpler but equivalent equation. We use the multiplication principle.

(continued on next page)

Replace 'X' by '$\frac{a}{2} + 2 + \frac{a}{6}$',

replace 'Y' by '$7 + \frac{a}{3}$',

and replace 'Z' by '6'.

Then, for every a,

$$\frac{a}{2} + 2 + \frac{a}{6} = 7 + \frac{a}{3}$$

if and only if

$$6\left(\frac{a}{2} + 2 + \frac{a}{6}\right) = 6\left(7 + \frac{a}{3}\right).$$

We simplify the members of the second equation to obtain:

$$3a + 12 + a = 42 + 2a.$$

Now, we solve this last equation. (Explain the steps.)

$$\begin{aligned} 4a + 12 &= 42 + 2a \\ 4a + 12 + (-2a) &= 42 + 2a + (-2a) \\ 2a + 12 &= 42. \end{aligned}$$

The final equation has the root 15. Let us see if 15 satisfies the original equation:

$$\frac{a}{2} + 2 + \frac{a}{6} = 7 + \frac{a}{3}$$

$\frac{15}{2} + 2 + \frac{15}{6}$	$7 + \frac{15}{3}$
$= 7\frac{1}{2} + 2 + 2\frac{1}{2}$	$= 7 + 5$
$= 12$	$= 12$

Do you see why we selected '6' to replace 'Z'? Look at the denominators of three algebraic fractions in the original equation. The number 6 is exactly divisible by 2, 3, and 6. It is called a <u>common multiple</u> of the numbers 2, 3, and 6.

Practice the method shown in Example 2 above by solving the following equations.

1. $\frac{x}{4} - 5 = \frac{x}{3}$
2. $\frac{t}{4} + \frac{t}{8} = 5 + \frac{t}{3}$
3. $\frac{3y}{4} + 6 = \frac{8y}{7}$
4. $\frac{k}{2} + \frac{3k}{5} - 1 = \frac{k}{10} - \frac{2k}{3}$

<u>Example</u> <u>3</u>. Solve the equation:

$$\frac{3}{t} - 1 = \frac{8}{3t} .$$

<u>Solution</u>. The denominators are 't' and '3t'. We see that for every t, 3t is a common multiple of t and 3t. We use the multiplication principle as follows:

Replace 'X' by '$\frac{3}{t} - 1$',

replace 'Y' by '$\frac{8}{3t}$',

and replace 'Z' by '3t'.

[Note that for every t, if $t \neq 0$, then $3t \neq 0$ and therefore, $Z \neq 0$.] The multiplication principle tells us that for every t, if $t \neq 0$, then

$$\frac{3}{t} - 1 = \frac{8}{3t}$$

if and only if

$$3t(\frac{3}{t} - 1) = 3t(\frac{8}{3t}) .$$

This last equation simplifies to:

$$9 - 3t = 8.$$

Now, solve this equation. It has the root $\frac{1}{3}$. The original equation should have the same root. Check by substituting '$\frac{1}{3}$' for 't' in the original equation:

$$\frac{3}{t} - 1 = \frac{8}{3t}$$

$\frac{3}{\frac{1}{3}} - 1$	$\frac{8}{3(\frac{1}{3})}$
$= 9 - 1$	$= \frac{8}{1}$
$= 8$	$= 8$

Practice the method shown in Example 3 above by solving the following equations.

1. $\frac{3}{a} - 2 = \frac{2}{a} + 3$

2. $\frac{4}{x} + \frac{5}{2x} = 3 - \frac{1}{x}$

3. $\frac{5}{2y} + \frac{1}{3y} = \frac{1}{y} + 2$

4. $\frac{7}{k} + \frac{1}{5} = \frac{2}{15k}$

<u>Example 4</u>. Solve: $5b = 15$

<u>Solution</u>. You can solve this equation immediately by the methods you used in the early part of this unit. But, you can also solve the equation by using the multiplication principle.

Replace 'X' by '5b',
replace 'Y' by '15',
and replace 'Z' by '$\frac{1}{5}$'.

Then, according to the principle, we know that for every b,

$$5b = 15$$

if and only if

$$\frac{1}{5}(5b) = \frac{1}{5}(15)$$

or, simply,

$$b = 3.$$

A root of the equation '$b = 3$' is, of course, 3. And 3 is also a root of the given equation.

Practice the method shown in the example above by solving the following equations even though you could solve them in other ways.

1. $27x = 810$

2. $4.26k = 0.003408$

3. $\frac{t}{83} = 97$

4. $\frac{s}{4.7} = 8.35$

* * *

The reasoning behind the Solution of Sample 1 may be a little easier to see if the set solution notation is used.

$$\{x:\ 2x + 3(5 - x) = 6(3 - x) + 5x\}$$

$$= \{x:\ -x + 15 = -x + 18\} \qquad [\text{p. e. a. e.}]$$

$$= \{x:\ -x + 15 + x = -x + 18 + x\} \qquad [\text{a. t. p.}]$$

$$= \{x:\ 15 = 18\} \qquad [\text{p. e. a. e.}]$$

$$= \emptyset$$

That $\{x:\ 15 = 18\} = \emptyset$ is easy to see if one thinks of the set selector '15 = 18' trying to find a directed number which will convert it into a true statement.

* * *

As an interesting exercise, ask students to solve the equation '5x = x' in two ways: by "common sense", and by using the transformation principles.

$$1 - \frac{2}{a} = 5$$

$$-\frac{2}{a} = 4$$

Therefore, the root is $-\frac{1}{2}$.

Such a student needs an equation like:

$$\frac{a - 2}{a + 3} = \frac{a - 7}{a + 4}$$

to cut his teeth on.

* * *

For your convenience we give the answers to all of the exercises. Students should check their answers. The practice in arithmetic is beneficial.

(1)	2	(2)	$\frac{1}{2}$	(3)	-1	(4)	$\frac{1}{2}$
(5)	1	(6)	3	(7)	5	(8)	-2
(9)	-3	(10)	$\frac{1}{2}$	(11)	-42	(12)	300
(13)	$\frac{63}{41}$	(14)	0	(15)	-1	(16)	$-\frac{1}{3}$
(17)	3.4	(18)	$\frac{19}{60}$	(19)	0	(20)	3

Note: In Exercises 13 and 18 (and in many others) a student may give a decimal name instead of a fractional name for a root. In some cases he will give an approximation. Naturally, he can expect only an "approximate" check when he gives an approximation to the root. We have no preference between decimal and fractional names for roots.

(continued on T. C. 37C)

You will want to spend considerable time on the 148 equations in this set of exercises. Assign them in small batches, checking each batch. Caution the students against getting shortcuts [transposing, for example] from "outside" sources. They will develop their own shortcuts, and this is to be expected and encouraged. But, the shortcuts should come as a result of the individual's own thinking and experience. Many of the equations in the set can be solved by the student's "common sense" methods rather than by application of the transformation principles. Do not insist upon his using a transformation principle when he can get the answer by a simpler procedure.

* * *

Some teachers reported vestiges of the "guessing" methods that the student used on equations earlier in the unit. If a student's guessing procedure shows any signs of intelligent thinking, then probably the desire to guess at a root is not something which should worry you. On the other hand, students should be convinced that the transformation principles are generally more powerful than their guessing procedures. One of the ways to convince the student of the superiority of the transformation principles is to let him use his guessing method in a race against a student who uses the transformation principles. If you find a student who, by guessing, consistently beats the student who uses the transformation principles, then you have a real discovery on your hands! For our own part, we think rather highly of a student who is expected (although not compelled) to use the multiplication transformation principle to solve the equation '$\frac{a - 2}{a} = 5$' but who actually solves it as follows:

(continued on T. C. 37B)

EXERCISES

Solve the following equations. Check your answers.

1. $4x + 3 = 2x + 7$
2. $6k - 2 = 8 - 14k$
3. $3 - 7r = 17 + 7r$
4. $8t - 3 = 12 - 22t$
5. $10k + 3 = 16 - 3k$
6. $4 - 7y = y - 20$
7. $7(x - 3) + 4 = 3x + 3$
8. $3(y - 2) + 5 = 3 + 5y$
9. $5(2x + 9) = 3(4x + 17)$
10. $9(3 - 2z) = 2(12 - 6z)$
11. $\frac{x}{3} - 2 = 5 + \frac{x}{2}$
12. $\frac{k}{4} - 3 = 12 + \frac{k}{5}$
13. $\frac{9y}{7} + 5 = 8 - \frac{2y}{3}$
14. $\frac{4m}{3} + \frac{3m}{5} = \frac{7m}{2}$
15. $\frac{8}{x} - 9 = \frac{17}{x}$
16. $\frac{3}{a} - 4 = \frac{7}{a} + 8$
17. $\frac{6}{5x} + 2 = \frac{8}{x}$
18. $\frac{7}{2b} - 6 = 9 - \frac{5}{4b}$
19. $\frac{1}{3}(5 - 7x) + \frac{1}{2}(4x + 7) = \frac{1}{6}(3x + 31)$
20. $\frac{2}{5}(8 - 3x) + \frac{5}{2}(6 - 7x) = \frac{3}{10}(4x + 15) - 46$

Sample 1. $2x + 3(5 - x) = 6(3 - x) + 5x$

Solution. First, simplify each member of the equation.

For every x,	For every x,
$2x + 3(5 - x)$	$6(3 - x) + 5x$
$= 2x + 15 - 3x$	$= 18 - 6x + 5x$
$= -x + 15$	$= -x + 18$

Thus, the given equation has the same roots as the equation:

$$-x + 15 = -x + 18.$$

We apply the addition principle.

For every x,

$$-x + 15 + x = -x + 18 + x$$

if and only if

$$15 = 18.$$

The statement '15 = 18' is false. Since the equation '15 = 18' is equivalent to the equation '-x + 15 = -x + 18',

(continued on next page)

(21) no roots	(22) 0	(23) no roots	(24) 2
(25) no roots	(26) 0		

* * *

In Sample 2, we note that

$$\{x:\ 4(x - 5) + 3x + 1 = 2(x - 2) + 5(x - 3)\} = \{x:\ -19 = -19\}.$$

Since the set selector '$-19 = -19$' is "converted" into a true statement by every number in the domain of 'x', the solution set of the given equation is the domain of 'x'. An equation in one pronumeral whose solution set is the domain of that pronumeral is called <u>an identity</u>. Thus, '$\frac{x}{x} = 1$' is an identity (with respect to the set of all non-zero directed numbers), and '$x = x$' is an identity (with respect to the set of all directed numbers).

$$\{x:\ \frac{x}{x} = 1\} \neq \{x:\ x = x\},$$

but $$\{x:\ x \neq 0 \text{ and } \frac{x}{x} = 1\} = \{x:\ x \neq 0 \text{ and } x = x\}.$$

we know that there is no number such that replacing 'x' by a numeral for that number makes '$-x + 15 = -x + 18$' a true statement. Therefore, since the original equation:

$$2x + 3(5 - x) = 6(3 - x) + 5x$$

is also equivalent to '$15 = 18$', it has no roots.

21. $7x - 2 = 5x + 2x$
22. $4x - 3 = 2x - 3$
23. $5(3 - x) - 1 = 13 - 5x$
24. $7(2 - x) + 4x = 2 + 3x$
25. $\frac{x}{3} + 4 = \frac{x}{2} - \frac{x}{6}$
26. $\frac{x}{7} + \frac{x}{3} = \frac{x}{21}$

Sample 2. $4(x - 5) + 3x + 1 = 2(x - 2) + 5(x - 3)$

Solution. Simplify each member of the equation.

For every x,	For every x,
$4(x - 5) + 3x + 1$	$2(x - 2) + 5(x - 3)$
$= 4x - 20 + 3x + 1$	$= 2x - 4 + 5x - 15$
$= 7x - 19$	$= 7x - 19$

Now, we solve the equation:

$$7x - 19 = 7x - 19$$

If we apply the addition principle to obtain an equivalent equation which has pronumerals in one member only, we get:

$$7x - 19 + (-7x) = 7x - 19 + (-7x)$$

or

$$-19 = -19.$$

The addition principle tells us that '$7x - 19 = 7x - 19$' and '$-19 = -19$' are equivalent equations. Since '$-19 = -19$' is true, then each replacement for 'x' in '$7x - 19 = 7x - 19$' must lead to a true statement. We conclude that every number is a root of the original equation:

$$4(x - 5) + 3x + 1 = 2(x - 2) + 5(x - 3).$$

Check by trying several replacements for 'x'. Notice that by applying the addition principle you can prove that every number is a root of this equation. You could not do this by substituting numerals for every number.

(27) all numbers	(28) all numbers	(29) 0.6
(30) all numbers	(31) all numbers	(32) 0
(33) -2.52	(34) 14	(35) $\frac{23}{4}$
(36) $\frac{11}{30}$	(37) -12	(38) $-\frac{4}{3}$
(39) 1.3	(40) 3.5	(41) 0.36
(42) 4.41	(43) -0.97	(44) 13.2
(45) 1520	(46) 152000	(47) -1
(48) 1	(49) $-\frac{1}{3}$	(50) $-\frac{1}{4}$
(51) 20	(52) $-\frac{8}{39}$	(53) $\frac{1}{2}$
(54) $\frac{5}{6}$	(55) $\frac{50}{3}$	(56) -10
(57) 7	(58) -2	(59) $-\frac{8}{3}$
(60) -4	(61) 0.9	(62) $\frac{8}{5}$
(63) $\frac{45}{16}$	(64) $\frac{1}{6}$	(65) 2
(66) 9	(67) 4	(68) 2
(69) -3	(70) -7	(71) $-\frac{2}{3}$
(72) no roots	(73) $-\frac{3}{7}$	(74) 4
(75) $\frac{50}{3}$	(76) $\frac{11}{4}$	

27. $4x + 4 = 4(x + 1)$

28. $8 - 5x + 6 = 7(2 + x) - 12x$

29. $5x - 3 = 3 - 5x$

30. $3x + 5 - x = 2(2 + x) + 1$

31. $\frac{x}{2} - \frac{x}{3} = \frac{x}{6}$

32. $\frac{x}{2} + \frac{x}{3} = \frac{x}{6}$

33. $y - 3.51 = -6.03$

34. $c - (-3) = 17$

35. $1 = R - \frac{19}{4}$

36. $x - \frac{1}{5} = \frac{1}{6}$

37. $-\frac{x}{2} = 6$

38. $-\frac{1}{3} = \frac{x}{4}$

39. $\frac{s}{10} = 13\%$

40. $\frac{3.5}{x} = 1$

41. $\frac{a}{0.3} = 1.2$

42. $-2.1 = \frac{b}{-2.1}$

43. $C + 113\% = 16\%$

44. $13 = z - 20\%$

45. $1\%A = 15.2$

46. $.01\%B = 15.2$

47. $4x + 7 = -3x$

48. $4y = y + 3$

49. $9a = 6a - 1$

50. $2x = 6x + 1$

51. $\frac{x}{4} - 3 = 2$

52. $10C = \frac{1}{4}C - 2$

53. $\frac{\frac{1}{2}R}{\frac{1}{2}} = \frac{1}{2}$

54. $\frac{\frac{2}{3}x}{\frac{1}{3}} = \frac{5}{3}$

55. $60\%z + 2 = 12$

56. $20\%t - 5 = -7$

57. $\frac{x + 3}{4} = 2.5$

58. $\frac{s}{4} = -\frac{1}{2}$

59. $-7 = 3x + 1$

60. $-3 = 2y + 5$

61. $\frac{2x}{9} = 20\%$

62. $\frac{x - 1}{3} = \frac{1}{5}$

63. $\frac{7}{4} = \frac{4}{9}x + \frac{1}{2}$

64. $\frac{3}{2} - x = \frac{4}{3}$

65. $3x + 5 - x - 8 = x + 7 - 4x$

66. $2(x - 3) + 6 = 3(2 - x) + 39$

67. $2(x - 3) + 3 = 5$

68. $2 - x = 2(x - 2)$

69. $-(2x + 3) + x = 0$

70. $7(y + 1) = 3(y - 7)$

71. $a + 1 + 2a = -(a + 1) + a$

72. $3(2z + 5) = 4(z - 5) + 2z$

73. $-3(x + 2) + (-4)(x - 1) = 1$

74. $-(4x - 1) = 2(x - 2.5) - 3(x + 2)$

75. $\frac{x}{2} - \frac{x}{5} = 5$

76. $6(x - 2x + 3) = \frac{3}{2}$

(continued on next page)

(77) 4	(78) 2	(79) 5
(80) $\frac{2}{5}$	(81) -18	(82) $-\frac{4}{35}$
(83) 0	(84) $-\frac{20}{3}$	(85) 0
(86) 8	(87) 1	(88) 1
(89) 15	(90) 1	(91) -2
(92) $\frac{1}{2}$	(93) $\frac{7}{4}$	(94) 0
(95) 0	(96) 6	(97) 0
(98) all numbers	(99) $\frac{5}{2}$	(100) 1
(101) 2	(102) all numbers	(103) 13
(104) -18	(105) 2	(106) 16
(107) no roots	(108) all numbers	(109) $-\frac{7}{2}$
(110) all numbers		
(111) 0	(112) -4	(113) $-\frac{16}{9}$
(114) $\frac{40}{3}$	(115) no roots	(116) 250
(117) -5	(118) $\frac{3}{17}$	(119) all numbers
(120) 10	(121) no roots	(122) no roots
(123) all numbers	(124) no roots	(125) no roots
(126) -4	(127) 0	(128) 20
(129) $\frac{250}{11}$	(130) 1	(131) $-\frac{1}{3}$
(132) $\frac{1}{3}$		

77. $2(x - 3) + 3(4 - x) = 2$

78. $a - 2(1 - a) = 7 - 3(a - 1)$

79. $\frac{1}{5}(3y - 12) = \frac{3}{5}$

80. $3a = \frac{1}{2}a + 1$

81. $25\%(L - 3) - 15\%(2L + 1) = 0$

82. $x + 2x + 0.5(x + 1) = 10\%$

83. $x = 2x - (x + 1) - x + 1$

84. $.32x + 0.4x(-3) = 6 + .02x$

85. $\frac{1}{2}x - \frac{1}{3}x = \frac{1}{12}x$

86. $\frac{4}{5}(t + 2) = t$

87. $\frac{4 - z}{6} = \frac{1}{2}$

88. $\frac{3(x + 1)}{2} = 3$

89. $2 + 2(b - 1) = b + 15$

90. $16y + 3(6y + 3) = 44 - y$

91. $1.5x = 3(x + 1)$

92. $-x - (x - 2) = 2x$

93. $6(2 - x) + x = 3x - 2$

94. $0.2t + 1 + 0.3t = 0.4t + 1$

95. $\frac{1}{3}(3x + \frac{x}{2}) = x$

96. $4(\frac{x}{3} + \frac{1}{2}) = \frac{x}{2} + 7$

97. $4a + 5 = 2(a + 2) + 1$

98. $x - (2x - 3) - (3 - x) = 0$

99. $2(x - 5) + 2x = 0$

100. $p + (p - 1) = 1$

101. $B - 2 = 2 - B$

102. $B \times 2 = 2 \times B$

103. $3(x - 3) = 2(x + 2)$

104. $2(6 - x) = -3(2 + x)$

105. $A + 2 = -4(1 - A)$

106. $C + 7 = 2C - 9$

107. $x = x + 5$

108. $2x - 3 = -(3 - 2x)$

109. $-6(x + 3) = 3$

110. $2.3s + 1.7s = 4s$

111. $-3m = m - (1 - m) - (3m - 1)$

112. $x + 1 + x + 2 = x + 3 + x + 4 + x$

113. $3x + .25x = x - 4$

114. $50\%x + 25\%x = 10$

115. $Z - 6 = Z + 17$

116. $10\%A + 5\%A = 25\%(150)$

117. $\frac{20\%R - R}{2} = 2$

118. $\frac{a + 3}{9} = \frac{3 - a}{8}$

119. $2(s + \frac{s}{2}) = 300\%s$

120. $\frac{L - 10}{15} = 0$

121. $2(P + 1) = 2(P - 1)$

122. $-4(L + 1) = 2(2 - 2L)$

123. $.03q + q = 100\%(q + 3\%q)$

124. $x + 15\%(2x + 1) = 130\%(x + 1)$

125. $\frac{s + 2}{12} = \frac{s}{3} - \frac{s}{4}$

126. $\frac{x + 0.5x}{2} = x + 1$

127. $\frac{L + 10\%L}{5} = L$

128. $\frac{6 - x}{2} = -7$

(continued on next page)

(133) $\frac{70}{13}$

(134) all numbers

(135) $\frac{3000}{617}$

(136) $-\frac{20}{3}$

(137) $\frac{4}{3}$

(138) 0.25

(139) all numbers

(140) $\frac{16}{1485}$

(141) 1

(142) $\frac{10}{11}$

129. $\frac{L + 10\%L}{5} = 5$

130. $\frac{x}{10} = \frac{x}{2} - \frac{x + 1}{5}$

131. $\frac{m + 3}{2} = \frac{m + 7}{5}$

132. $\frac{4a + 1}{3} = \frac{5 - a}{6}$

133. $0.3x = 7 - x$

134. $6 - 2n = -n - (n - 6)$

135. $6x + 17\%x = 30$

136. $3x - 1x = 5(x + 4)$

137. $\frac{x}{2} - \frac{x}{4} + \frac{2 - 3x}{6} = 0$

138. $\frac{m + 2}{3} = \frac{m + .5}{1}$

139. $1.6(x + 1) = -2(x + 1) + 3.6(x + 1)$

140. $10\%x + 150x = 1.6(x + 1)$

141. $4(R + 1.1) = 2(2 - 2R) + 840\%$

142. $11\%y - 20\%y = 90\%(y - 1)$

Sample 3. $\frac{2}{a - 3} = \frac{5}{a}$

Solution. We note that for every a, a common multiple of a - 3 and a is a(a - 3). We apply the multiplication principle.

Replace 'X' by '$\frac{2}{a - 3}$',

replace 'Y' by '$\frac{5}{a}$',

and replace 'Z' by 'a(a - 3)'.

Since 'a(a - 3)' stands for 0 when 'a' is replaced by either a numeral for 0 or a numeral for 3, we may state:

For every a, if $a \neq 0$ and $a \neq 3$,

then

$$\frac{2}{a - 3} = \frac{5}{a}$$

if and only if

$$a(a - 3)\frac{2}{a - 3} = a(a - 3)\frac{5}{a}$$

or, simply,

$$2a = 5(a - 3).$$

Solve the equation:

$$2a = 5(a - 3)$$
$$2a = 5a - 15$$
$$2a + (-5a) = 5a - 15 + (-5a)$$
$$-3a = -15$$
$$a = 5.$$

Students in Miss Blair's class made the following point with regard to Exercises 10 through 13:

> Since '25%' is another name for 0.25, then the blank in Exercise 10 should be replaced by 40.25.

This observation is correct. Our intention, however, was that the blank should be replaced by '50'. We were using the expression '25% greater than 40' to mean the same as '25% of 40 greater than 40'. This is the common, non-technical usage of that expression. You should point out to your students that although this is common usage, if they interpret the expression in the strict technical sense, they should come to the conclusion that Miss Blair's students did. But, they should understand "everyday" usage also.

* * *

For all the instances to make sense in Exercise 20 on page 3-43, you should rewrite it as:

For every $x \neq 0$, the quotient of $7x$ by $2x$ is ___________ .

Method 2.

$$\frac{5}{x - 1} = \frac{5}{1 - x}$$

$$\frac{5}{x - 1} = \frac{-5}{x - 1} \qquad \text{[p.e.a.e.]}$$

$$\frac{5}{x - 1}(x - 1) = \frac{-5}{x - 1}(x - 1), \quad x \neq 1 \qquad \text{[m.t.p.]}$$

$$5 = -5 \qquad \text{[p.e.a.e.]}$$

The given equation has no roots.

$$\{x:\ x \neq 1 \text{ and } \frac{5}{x - 1} = \frac{5}{1 - x}\} = \{x:\ x \neq 1 \text{ and } 5 = -5\}$$
$$= \emptyset .$$

Exercise 148.

Formal methods are applied as follows:

$$\frac{2}{x - 3} = \frac{2}{x - 3}$$

$$\frac{2}{x - 3}(x - 3) = \frac{2}{x - 3}(x - 3), \quad x \neq 3 \qquad \text{[m.t.p.]}$$

$$2 = 2 \qquad \text{[p.e.a.e.]}$$

Each directed number except 3 is a root of the given equation.

$$\{x:\ x \neq 3 \text{ and } \frac{2}{x - 3} = \frac{2}{x - 3}\} = \{x:\ x \neq 3 \text{ and } 2 = 2\}$$
$$= \{x:\ x \neq 3\}$$

* * *

The Exploration Exercises are preparation for the worded problems which follow in the next section. As usual, if a student has difficulty completing a statement, he should first do the exercise by replacing the pronumeral by a numeral.

* * *

(continued on T. C. 42C)

(143) 9   (144) 10   (145) $\frac{1}{4}$

(146) $\frac{16}{9}$ (147) no roots   (148) all numbers except 3

<u>Exercise</u> <u>147</u>.

<u>Method</u> <u>1</u>.

$$\frac{5}{x - 1} = \frac{5}{1 - x}$$

$$\left[\frac{5}{x - 1}\right](x - 1)(1 - x) = \left[\frac{5}{1 - x}\right](x - 1)(1 - x), \quad [x \neq 1] \qquad \text{[m.t.p.]}$$

$$5(1 - x) = 5(x - 1) \qquad \text{[p.e.a.e.]}$$

$$5 - 5x = 5x - 5 \qquad \text{[p.e.a.e.]}$$

$$5 - 5x + 5x = 5x - 5 + 5x \qquad \text{[a.t.p.]}$$

$$5 = 10x - 5 \qquad \text{[p.e.a.e.]}$$

$$5 + 5 = 10x - 5 + 5 \qquad \text{[a.t.p.]}$$

$$10 = 10x \qquad \text{[p.e.a.e.]}$$

$$10 \times \frac{1}{10} = 10x \times \frac{1}{10} \qquad \text{[m.t.p.]}$$

$$1 = x \qquad \text{[p.e.a.e.]}$$

Since our first step excluded 1 from the domain of 'x', this last equation has no roots. Hence the given equation has no roots. That is,

$$\{x\colon\ x \neq 1 \text{ and } \frac{5}{x - 1} = \frac{5}{1 - x}\} = \{x\colon\ x \neq 1 \text{ and } x = 1\}$$
$$= \emptyset .$$

(continued on T. C. 42B)

Hence, the root of '2a = 5(a - 3)' is 5. Since neither 0 nor 3 is a root of:

$$2a = 5(a - 3)$$

this equation is equivalent to the original equation. Therefore, the original equation:

$$\frac{2}{a - 3} = \frac{5}{a}$$

must have the root 5. Does it? Check by substituting.

143. $\frac{7}{x - 2} = \frac{9}{x}$

144. $\frac{2}{x + 4} = \frac{1}{x - 3}$

145. $\frac{3}{2x + 1} = \frac{8}{4x + 3}$

146. $\frac{4}{2 - x} = \frac{6}{3x - 5}$

147. $\frac{5}{x - 1} = \frac{5}{1 - x}$

148. $\frac{2}{x - 3} = \frac{2}{x - 3}$

EXPLORATION EXERCISES

Use the simplest expression you can to complete each of the following statements correctly.

1. The sum of 5 and 9 is ______.
2. For every x, the sum of x and x + 7 is ______.
3. For every x, the sum of x and 3x - 2 is ______.
4. For every x, a number 5 times as large as x is ______.
5. For every x, the difference of x from a number 6 times as large as x is ______.
6. For every $\square$, the number which is 2 greater than $\square$ is ______.
7. The number which is 4% of 30 is ______.
8. For every x, the number which is 10% of x is ______.
9. For every Δ, the number which is 15% of 7Δ is ______.
10. The number which is 25% greater than 40 is ______.
11. For every P, the number which is 20% greater than P is ______.
12. The number which is 70% less than 70 is ______.
13. For every R, the number which is 80% less than R is ______.
14. For every S and T, the sum of 50% of S and 60% of T is ______.

(continued on next page)

15. For every A, the product of $\frac{1}{2}$A and 150 is _______.

16. The number which is 5 less than 12 is _______.

17. For every z, the number which is 9 less than z is _______.

18. The number by which 73 exceeds 62 is _______.

19. For every v, the number by which 3v exceeds v + 2 is _______.

20. For every x, if $x \neq 0$, then the quotient of 7x by 2x is _______.

21. You can walk _______ miles in 3 hours at the average rate of 4 miles per hour.

22. For every h, you can walk _______ miles in h hours at the average rate of 4 miles per hour.

23. For every r, you can travel _______ miles in 3 hours at the average rate of r miles per hour.

24. For every h and r, you can travel _______ miles in h hours at the average rate of r miles per hour.

25. If John is 12 years old, then he was _______ years old 5 years ago.

26. For every x, if Bill is x years old now, then he was _______ years old 6 years ago.

27. For every y, if Jack is 2y years old at present, then he will be _______ years old 7 years from now.

28. For every z, if Mary is now z years old and if Bill's age now is 3 years less than twice Mary's age, Bill is now _______ years old.

29. For every x, if the difference between Jim's and Andy's ages is x years and if both Jim and Andy are now at least 4 years old, then the difference between their ages 4 years ago was _______ years.

30. A loaf of white bread costs 3 cents less than a loaf of whole wheat bread. Three loaves of whole wheat bread cost _______ cents more than three loaves of white bread.

31. A pad of lined paper costs 5 cents more than a pad of unlined paper. For every x, if unlined paper costs x cents a pad, the cost of 6 pads of lined paper is _______ cents.

(continued on next page)

32. There are ______ pints in 3 quarts.

33. For every k, there are ______ pints in k quarts.

34. For every m, there are ______ pints in a total of m quarts and 2m pints.

35. There are ______ feet in 24 inches.

36. For every t, there are ______ feet in t inches.

37. For every x, there are ______ feet in a total length of 2x yards, 4x feet, and 7x inches.

38. There are ______ cents in 7 nickels.

39. For every m, there are ______ cents in m nickels.

40. For every p and t, there are ______ cents in a total of p nickels and t dimes.

41. For every d, there are ______ dollars in d dimes.

42. A paper boy earns one cent for each paper delivered. If every day except Sunday, he delivers papers to 35 customers, then he earns ______ dollars per week.

43. For every x, the paper boy, mentioned in Exercise 42, earns ______ dollars per week if he has x customers.

44. Suppose the paper boy, mentioned in Exercise 42, also delivers papers on Sunday. Then, for every y, he earns ______ dollars per week if he gets 2 cents for each Sunday paper delivered and if he has y customers for each day of the week including Sunday.

45. John buys pencils at the rate of 3 pencils for a dime. If he sells them to his classmates at 5 cents each, his profit on the sale of 18 pencils is ______ cents.

46. For every p, John's profit (Ex. 45) on the sale of 3p pencils is ______ cents.

47. The cost price of an article is $25. The margin is $18. The selling price is ______ dollars.

48. For every x, if the cost price of an article is x dollars and the margin is 20% of the cost price, then the selling price is ______ dollars.

(continued on next page)

49. For every w, if the width of a rectangle is w inches and the length of the rectangle is twice the width, then the perimeter is _____ inches.

50. For every x, the perimeter of a square is ______ inches if the length of a side is 3x inches.

51. It takes ______ hours for a train to travel 80 miles if its average rate is 60 miles per hour.

52. For every x, it takes ______ hours for a freight train to travel x miles if its average rate is 30 miles per hour.

53. For every s, you must walk at an average rate of ______ miles per hour to travel 4s miles in 2 hours.

54. The interest per year on $700 invested at an annual rate of 4% is ______ dollars.

55. The annual income (interest) on $2000 invested at 5.5% is _____ dollars.

56. For every x, the annual income on x dollars invested at 3% is _____ dollars.

57. For every y, if $y \leq 1400$, the annual income on 1400 - y dollars invested at 4.5% is ______ dollars.

58. For every s, if the length of each side of a regular pentagon is 2s + 7 inches, the perimeter is ______ inches.

59. For every x, if the length of each of the two equal sides of an isosceles triangle is 2 inches longer than the length of the base and if the length of the base is x inches, then the perimeter is ______ inches.

60. For every t, if the circumference of a circle is 33t inches, the diameter is ______ inches.

61. For every k, the area of the surface of a cube each of whose edges is 7k inches is ______ square inches.

62. For every x, there are ______ cents in a total of x nickels and 2x + 3 dimes.

63. For every t, there are _____ quarts in a total of t pints and 3t quarts.

64. ______ is 125% of 60.

65. For every A, ______ is 15% of 3A.

(continued on next page)

Use Exercise 76 to lead into a more intensive treatment of "mixtures". The following exercise proved helpful in one of our schools.

If a man starts with △ quarts of ○ % alcohol and adds □ quarts of ⬡ % alcohol, then he has ⬭ quarts of ▭ % alcohol.

	△	○	□	⬡	⬭	▭
(1)	5	30	15	20		
(2)	10	25		15		18
(3)	15	30	20			50
(4)		100	40	30		23.5
(5)	16		30	45		70

Construct similar exercises for the candy mixtures [Exercise 77] and the nut mixtures [Exercise 78].

66. The total cost of 10 pounds of coffee at 85 cents per pound and 7 pounds of coffee at 72 cents per pound is ______ dollars.

67. For every x, if $x \leq 27$, the total cost of x pounds of coffee at 93 cents per pound and 27 - x pounds of coffee at 89 cents per pound is ______ dollars.

68. ______ is a number with a two-digit numeral whose tens digit is '7' and whose units digit is '3'.

Note: In Exercises 69 and 70 the pronumerals are replaceable by digits ('0', '1', '2', '3', '4', '5', '6', '7', '8', or '9') only.

69. For every x, if x is a whole number and $0 < x < 10$, then ______ is a number with a two-digit numeral whose tens digit is 'x' and whose units digit is 'x + 4'.

70. For every x and y, if x and y are whole numbers and $0 < x < 10$ and $0 < y < 10$, then ______ is a number with a two-digit numeral whose tens digit is 'x' and whose units digit is 'y'.

71. For every n, if n is a whole number, then ______ is the next larger whole number.

72. For every n, if n is an odd whole number, then ______ is the largest whole number smaller than n.

73. For every m, if m is a whole number, then the sum of the next two consecutive whole numbers is ______.

74. If a man can do a certain job is 12 minutes, then by working at the same rate he can do ______ of the job in 1 minute, ______ of the job in 2 minutes, ______ of the job in 9 minutes, and ______ of the job in 12 minutes.

75. For every x, if a man can do a certain job in 10 minutes, then by working at the same rate he can do ______ of the job in x minutes.

76. If 25% of 4 quarts of an alcohol solution is alcohol, then adding 2 quarts of pure alcohol to the solution gives a new solution of which ______ is alcohol.

77. If 50% of 3 pounds of a candy mixture is peppermint candy, then to make a mixture which contains 80% peppermint candy you must add ______pounds of peppermint candy to the original mixture.

(continued on next page)

of the more recent ones) are full of teaching devices which succeed in getting students to solve certain kinds of worded problems in a mechanical way. There is very little to be gained by using such techniques. They do nothing whatsoever to increase the student's power in mathematics; they simply make an "expert" of him in solving restricted types of worded problems. If this expert is given a worded problem which varies only slightly from the type he has learned to solve mechanically, then he is completely lost.

There may be some students in your class who can solve the 47 problems in this section in three to five days of assignments. There may be others for whom these problems provide enough work for $1\frac{1}{2}$ months of assignments. We suggest that you differentiate among your students as far as these problems are concerned. Mr. Dietz and Mr. Marston did this by spreading the problems (and others like them which you can find in any elementary algebra book) over a considerable period of time, assigning one or two problems each day and checking the work individually.

Be sure to indicate in your weekly reports the progress you are making with worded problems.

The article may be more readily accessible to you in its reprinted form in Volume I of The World of Mathematics (pages 170-178) recently published by Simon and Schuster and edited by Newman.]

* * *

Teachers last year had difficulty in getting some of their students to make the transition from the "guessing method" to the "pronumeral method". Part of the difficulty, we believe, can be attributed to the word 'guessing'. Actually, we are not concerned here with obtaining an intelligent guess in solving a worded problem. [If a student can make intelligent guesses, then he is probably the kind of student who can use the pronumeral method whenever he is called upon to do so.] Instead of a guess, what we want is a sample answer. Then, checking the sample answer against the problem itself and using the pronumeral frames as we illustrate should result in the student's constructing the required equation. In order to obtain this equation, all the student needs to do is to erase the sample answer from the pronumeral frames. If he wants to substitute 'x' or some other letter in place of a frame type pronumeral he may do so.

What needs to be stressed is the fact that we want to obtain an equation containing a pronumeral, rather than trying to guess at the correct answer.

As we indicated in the first comment on T. C. 47A, there is no easy road to solving worded problems. We think that a student's facility in solving problems is highly correlated with his native intelligence. All of us know that it is possible to get students to solve certain kinds of worded problems in a fairly mechanical manner. Elementary algebra textbooks of 30 years ago (and some

(continued on T. C. 47C)

As far as we know there is no universal formula for solving worded (or verbal) problems in mathematics. It appears that the most effective teaching technique is to give the students plenty of practice. We urge that you do not insist upon excessive formality in the student's work. If you require a certain amount of description to be written by the student on his homework paper or on his test paper so that you have something to go on in grading him or in pointing out his errors, we suggest that you inform him of your purposes in requiring this description of his procedure.

Our three illustrations are long-winded. The technique used is relatively new and requires detailed explanation on our part. We do not expect students to write similar explanations when they solve problems on their own.

* * *

The article by Henderson and Pingry on problem-solving in the 21st Yearbook of the National Council of Teachers of Mathematics contains a number of helpful suggestions. Of particular importance is the research they summarize regarding the formation of mental sets. In order to avoid formation of inhibitory mental sets, we have not "typed" our problems.

* * *

The ancient manuscript to which we refer is the Rhind Papyrus. Most recent histories of elementary mathematics include a discussion of the Rhind Papyrus. [See, for example, Howard Eves, An Introduction to the History of Mathematics (New York: Rinehart and Company, 1953) p. 46. James R. Newman has an excellent article on the Rhind Papyrus in the August 1952 issue of Scientific American.

(continued on T. C. 47B)

78. For every x, if x ounces of shelled peanuts are added to 2 pounds of nut mixture which contains 30% shelled peanuts, the new mixture contains _______ pounds of shelled peanuts.

3.06 Using pronumerals to solve problems.--Now that you have learned how to solve equations, you are ready to solve problems like those that were given at the beginning of this unit. Problems like these are commonly found in high school mathematics--in fact, problems of this kind have been found in manuscripts that are thousands of years old. Most students like these problems because it is interesting to puzzle them out. Although the problems deal with such things as coins, tickets, mixtures, interest, rates, areas, perimeters, and other things that are of a practical nature, you should realize that these problems are really only puzzles. The problems themselves may not be practical, but it is important that you develop methods for solving them and that you practice using these methods. Such practice will help you solve more important and practical problems in the future.

If you want to become a good problem-solver, you will have to develop your own ways of attacking a problem. We shall give you samples of how some problems can be attacked and solved. You may find these illustrations helpful in developing your own methods.

Problem 1. A jar of coins contains 3 times as many dimes as nickels and twice as many quarters as nickels. The total value of the quarters, dimes, and nickels in the jar is $21.25. How many nickels are there in the jar?

A good way to attack the problem and to see that you really understand it is to make a guess at the answer, and then to check your guess. Suppose we guess that there are 10 nickels in the jar. Now, we go through the problem using our guess.

A jar of coins contains 3×10 dimes, 2×10 quarters, and 10 nickels. The total value of the quarters, dimes, and nickels in the jar is found in the following way:

2×10 quarters gives $25 \times 2 \times 10$ or 500 cents;

3×10 dimes gives $10 \times 3 \times 10$ or 300 cents;

10 nickels gives 5×10 or 50 cents;

the total value is

$$500 + 300 + 50 \text{ or } 850 \text{ cents.}$$

Since we are told that the value of the coins is $21.25 or 2125 cents, our guess leads to the statement:

$$850 = 2125.$$

But, this last statement is false. So, our guess was wrong. (It was too low.) Now, we could continue to make guesses and, eventually, we would hit upon the right answer. (In some problems the "guess" method might not ever lead you to a correct answer.) There is another way to attack this problem which involves less work.

If you were to make another guess and work through the problem using the guess, the actual arithmetic would be quite like the arithmetic we did in checking your guess above. The arithmetic would follow the same pattern. If we follow this pattern with a pronumeral rather than with a numeral, we can check a guess merely by putting a numeral in place of the pronumeral. Look at the way in which we used the numeral '10' when we checked our guess above. Instead of using a numeral, let us use a pronumeral say, '$\square$', and go through the problem again.

A jar of coins contains 3$\square$ dimes, 2$\square$ quarters, and $\square$ nickels. The total value of the quarters, dimes, and nickels is found in the following way:

$$2\square \text{ quarters gives } 25 \times 2\square \text{ or } 50\square \text{ cents;}$$

$$3\square \text{ dimes gives } 10 \times 3\square \text{ or } 30\square \text{ cents;}$$

$$\square \text{ nickels gives } 5\square \text{ cents;}$$

the total value is

$$50\square + 30\square + 5\square \text{ or } 85\square \text{ cents.}$$

Since we are told that the value of the coins is 2125 cents, we write:

$$85\square = 2125.$$

A student might solve Problem 1 by writing merely the following:

$$\square \text{ nickels}$$
$$3\square \text{ dimes}$$
$$2\square \text{ quarters}$$

$$5\square + 30\square + 50\square = 2125$$
$$85\square = 2125$$
$$\square = 25$$

This abbreviated description is quite satisfactory and just as clear as:

Let $\square$ = the number of nickels,
then $3\square$ = the number of dimes, etc.,

although the latter is acceptable.

The last expression is an equation in the pronumeral '$\boxed{}$'. If we can find a root of this equation, and put a numeral for the root in every '$\boxed{}$' above, then we shall have hit upon a correct "guess". Thus, finding the answer to the problem is the same as solving the equation '$85\,\boxed{} = 2125$'. A root of this equation is 25. We write '25' in each '$\boxed{}$' and go through the problem again.

> A jar of coins contains $3 \times \boxed{25}$ dimes, $2 \times \boxed{25}$ quarters, and $\boxed{25}$ nickels. The total value of the quarters, dimes, and nickels is found in the following way:
>
> $2 \times \boxed{25}$ quarters gives $50 \times \boxed{25}$ or 1250 cents;
>
> $3 \times \boxed{25}$ dimes gives $30 \times \boxed{25}$ or 750 cents;
>
> $\boxed{25}$ nickels gives $5 \times \boxed{25}$ or 125 cents;
>
> the total value is
>
> 1250 + 750 + 125 or 2125 cents or $21.25.

So, we know that there are 25 nickels in the jar. The problem is solved.

<u>Problem 2</u>. A committee sold 120 tickets for a school play. Some of the tickets were sold to adults at $.70 each; the reamining tickets were sold to students at $.50 each. A total of $70.40 was collected from ticket sales. How many adult tickets were sold?

Let us take a guess at the answer. Suppose 40 adult tickets were sold. Now, check this guess.

> A committee sold 120 tickets. 40 tickets were sold to adults at $.70 each; 120 - 40 were sold to students at $.50 each. The total number of cents collected is obtained as follows:
>
> 40 tickets @ $.70 gives 70×40 or 2800 cents;
>
> 120 - 40 tickets @ $.50 gives $50 \times (120 - 40)$ or 4000 cents;
>
> the total number of cents collected is
>
> 2800 + 4000 or 6800.
>
> Since we are told that the total collected is $70.40, our guess leads us to write:
>
> 6800 = 7040.

The last statement is false. Therefore, we know that our guess was wrong.

Now, instead of guessing again, let us set up a pattern which could be used for checking any guess by going through the problem using a pronumeral.

A committee sold 120 tickets. x tickets were sold to adults at \$.70 each; 120 - x were sold to students at \$.50 each. The total number of cents collected is obtained as follows:

x tickets @ \$.70 gives 70x cents;
120 - x tickets @ \$.50 gives 50(120 - x) cents;

the total number of cents collected is

$$70x + 50(120 - x).$$

Since we are told that 7040 cents were collected, we write:

$$70x + 50(120 - x) = 7040.$$

This last expression, of course, is neither true nor false. It simply gives us a way of checking guesses. All we need to do now is to solve the equation:

$$70x + 50(120 - x) = 7040.$$

We know that when a numeral for the root replaces 'x' in this equation, we get a true statement. Therefore, the root of the equation gives us an answer to our problem. So, we solve the equation.

$$\begin{aligned} 70x + 50(120 - x) &= 7040 \\ 70x + 6000 - 50x &= 7040 \\ 20x + 6000 &= 7040 \\ 20x &= 1040 \\ x &= 52 \end{aligned}$$

A root of the equation is 52. Therefore, we conclude that 52 adult tickets were sold. Now, let us check this answer.

52 adult tickets @ \$.70.....	\$36.40
120 - 52 or 68 student tickets @ \$.50.....	34.00
	\$70.40

Problem 3. John usually takes 40 minutes to ride his bicycle from home to school. When he is pressed for time, he can increase his average speed by 6 miles per hour and save 16 minutes. How far does John live from school?

Let us guess that John lives 5 miles from school. We check this guess.

Usual time for the 5-mile trip is 40 minutes or $\frac{2}{3}$ hours.

Usual average speed is $\frac{5}{\frac{2}{3}}$ or $\frac{5 \times 3}{2}$ miles per hour.

If he increases his usual speed by 6 miles per hour, then his new speed is $\frac{5 \times 3}{2} + 6$ miles per hour. At this speed he can make the 5-mile trip in $\frac{5}{\frac{5 \times 3}{2} + 6}$ hours. We are told that the trip at the new speed takes 40 - 16 or 24 minutes which is $\frac{2}{5}$ hours. So our guess leads us to write:

$$\frac{5}{\frac{5 \times 3}{2} + 6} = \frac{2}{5} .$$

This statement is false because the expression at the left of '=' simplifies to '$\frac{10}{27}$'. Now, we go through the problem using a pronumeral.

Usual time for the d-mile trip is $\frac{2}{3}$ hours. Usual average speed is $\frac{d}{\frac{2}{3}}$ or $\frac{3d}{2}$ miles per hour. The increased speed is $\frac{3d}{2} + 6$ miles per hour. The trip should take $\frac{d}{\frac{3d}{2} + 6}$ hours. Therefore, we can write:

$$\frac{d}{\frac{3d}{2} + 6} = \frac{2}{5} .$$

Now, we solve this equation.

$$\left[5\left(\frac{3d}{2} + 6\right)\right] \times \left(\frac{d}{\frac{3d}{2} + 6}\right) = \left[5\left(\frac{3d}{2} + 6\right)\right] \times \left(\frac{2}{5}\right)$$

(continued on next page)

(c) <u>a straight-forward arithmetic method</u>

115 3's are worth 345 cents which is 83 cents more than she spent. Since the difference in value between a 3-cent stamp and a 2-cent stamp is 1 cent, the 83-cent difference can be made up by "changing" 83 3's into 83 2's

Note that method (c) actually involves the steps taken in solving the equation obtained by the pronumeral method:

$$3(115 - x) + 2x = 262$$

$$3(115) - 3x + 2x = 262$$

$$345 - 262 = (3 - 2)x$$

$$83 = x$$

It should be clear to the student that you are trying to teach him method (a). Moreover, it ought to be the case that each of method (b) and method (c) takes longer than (a). If a student can demonstrate to you that he is consistently successful and faster with either (b) or (c) than with (a), then either our problems are too easy, or the method (a) is indeed a less efficient method.

We illustrate below with Exercise 1, (a) the use of a sample answer to obtain an equation (the pronumeral method), (b) a non-algebraic procedure based on an intelligent guess and, (c) the use of a straightforward arithmetic procedure.

(a) the pronumeral method

Suppose Agnes bought 100 2-cent stamps. Then she must have (115 - 100) 3-cent stamps. The 2-cent stamps are worth 2×100 cents and the 3-cent stamps are worth $3(115 - 100)$ cents. So, altogether, the stamps are worth

$$2 \times 100 + 3(115 - 100) \text{ cents}$$

In other words, if our sample answer happens to be correct, the equation:

$$2 \times 100 + 3(115 - 100) = 262$$

should be a correct statement. Now, we don't care whether this equation is a true statement. All we care about is the fact that the correct answer must be a root of:

$$2 \times \bigcirc + 3(115 - \bigcirc) = 262$$

So, solve this equation as you did earlier in Unit 3.

(b) an intelligent guess method

We know that Agnes bought less than 115 2-cent stamps. Let us say she bought 100 of them. Then, she must have bought 15 3-cent stamps. But, 100 2's and 15 3's are worth $2.45 which is 17 cents less than she actually spent. So, if we "change" 17 2's into 17 3's, we shall make up this difference. Therefore, she bought 83 2's and 32 3's.

(continued on T. C. 52C)

Give an alternative solution to Problem 3. Instead of using a pronumeral to set up a pattern for checking guesses about the distance John lives from school, use a pronumeral for checking guesses about John's usual rate.

x miles per hour is usual rate

$\frac{2}{3}x$ miles is distance from home to school

$x + 6$ miles per hour is increased rate

$\frac{2}{5}(x + 6)$ miles is distance from home to school

$$\frac{2}{3}x = \frac{2}{5}(x + 6)$$

$$15\left(\frac{2}{3}x\right) = 15\left[\frac{2}{5}(x + 6)\right]$$

$$10x = 6(x + 6)$$

$$10x = 6x + 36$$

$$4x = 36$$

$$x = 9$$

Distance from home to school if $\frac{2}{3}(9)$ or 6 miles.

Students should understand that there are often several ways of solving a given problem and that one of the wyas may lead to a very simple equation. The ability to hit upon the most elegant way comes as a result of extended experience; maybe it is one of those skills that can be learned but not taught!

* * *

The first five or six exercises are more difficult than one would expect to find at the beginning of a long set of exercises. We want the student to ''cut his teeth'' on a few difficult problems to impress him with the idea that the pronumeral method is not the teacher's ''hard way to do an easy task''.

(continued on T. C. 52B)

$$[5] \times (d) = \left[\frac{3d}{2} + 6\right] \times (2)$$

$$5d = 3d + 12$$

$$2d = 12$$

$$d = 6$$

[Note that since 6 is the only root of the equation 'd = 6', and since when 'd' is replaced by '6' in the expression '$5\left(\frac{3d}{2} + 6\right)$' we do not get an expression for 0, the multiplication principle was applied correctly. Therefore, a root of the original equation is 6.]

Thus, we conclude that John lives 6 miles from school. We check this answer.

> Usual average speed is $\frac{6}{\frac{2}{3}}$ or 9 miles per hour. Increased speed is 9 + 6 or 15 miles per hour. New time of travel is $\frac{6}{15}$ hours or 24 minutes which is 16 minutes less than 40 minutes.

In solving the first few problems which follow you should guess at an answer, check your guess, and then use a pronumeral in the same manner as we did in the problems above. After a while you will be able to use the pronumeral method without guessing at all. Perhaps some problems will be so easy for you that you will not need to use the pronumeral method. However, you may want to practice the pronumeral method even on the easy problems so that you will become skillful in its use.

EXERCISES

Solve these problems.

1. Agnes bought 115 stamps for $2.62. Some of the stamps were 2-cent stamps and the others were 3-cent stamps. How many 2-cent stamps did she buy?

2. One number is 12 less than another number. The sum of the two numbers is 93. What is the smallest number?

(continued on next page)

[illegible]

answer.

[illegible]

PROBLEMS

Solve these problems.

1. [illegible]

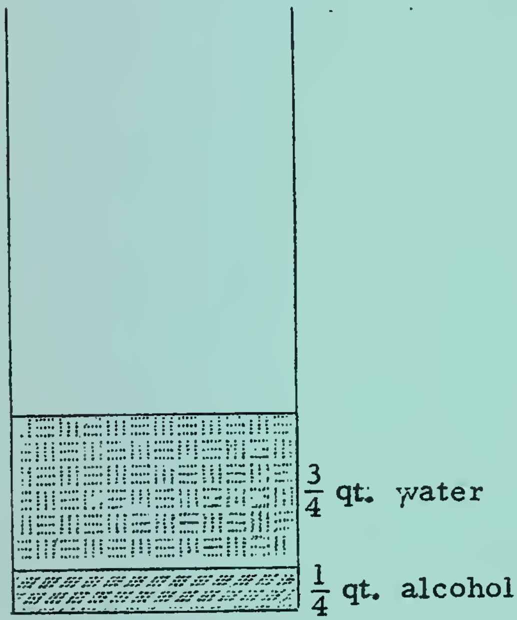
2. [illegible]

(continued on next page)

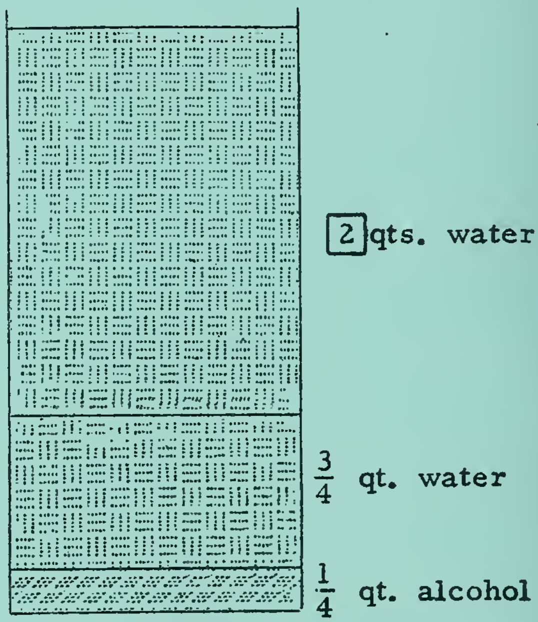
[illegible]

The "mixture" problems illustrated in Exercises 11 and 12 are usually difficult. [Mr. Dietz demonstrated with liquids and containers to give students a feeling for this kind of problem.] Here is a device we have used.

Consider Exercise 11. A sample answer is $\boxed{2}$ quarts.



Original Mixture



New Mixture

In the new mixture there are $\boxed{2} + 1$ quarts of liquid and $\frac{1}{4}$ of a quart of alcohol. Therefore, '$\frac{1}{4} = \frac{1}{5}(\boxed{2} + 1)$' is true if our sample answer happens to be correct. But we don't care to check this. We do know, however, that the <u>correct</u> answer to our problem can be found by solving the equation:

$$\frac{1}{4} = \frac{1}{5}(\square + 1)$$

or, for easier writing:

$$\frac{1}{4} = \frac{1}{5}(x + 1)$$

3. Mary is 3 years older than Bill. Ten years ago Mary was three times as old as Bill. How old is Mary now?

4. A rectangle is twice as long as it is wide. If each dimension were increased by 3 inches, the new perimeter would be three times the old perimeter. What are the dimensions of the rectangle?

5. Mrs. Adams paid \$9.35 for a train ticket. This price included 10% Federal tax. How much was the tax?

6. I am thinking of a certain number. Taking $\frac{1}{5}$ of the number gives the same result as subtracting the number from 27. What is the number?

7. A square and a rectangle have equal perimeters. The rectangle is 22 feet by 36 feet. What is the length of a side of the square?

8. The sum of two numbers is $112\frac{3}{4}$. The difference of the smaller from the larger is $5\frac{3}{4}$. What are the two numbers?

9. One number is 7 more than another number. The smaller number is 44% of the larger number. What are the two numbers?

10. The sum of two consecutive whole numbers is 1003. What is the smaller of these numbers?

11. One quart of an alcohol solution contains 25% alcohol and 75% water. How much water should be added to make a solution which is 20% alcohol?

12. One pint of an alcohol solution contains 15% alcohol and 85% water. How much alcohol must be added to make a solution which contains 35% alcohol?

13. The sum of three consecutive whole numbers is 180. What are the numbers?

14. A salesman works on a "base pay plus commission" salary arrangement. Suppose that his base pay is \$125 per month and that he gets a 4% commission on his sales. How much must he sell to earn \$500 per month?

15. John has a handful of dimes and nickels totaling \$3.55. The number of dimes is 7 greater than the number of nickels. How many dimes does John have?

(continued on next page)

You will be interested in the following arithmetic solution one student devised for Exercise 26:

> Pipe I can fill the tank in 17 minutes. It could fill 21 such tanks in 21×17 minutes.
>
> Pipe II can fill the tank in 21 minutes. It could fill 17 such tanks in 17×21 minutes.
>
> Then in 21×17 minutes, pipe I fills 21 tanks and pipe II fills 17 tanks. That is, by working together, both pipes can fill $17 + 21$ tanks in 17×21 minutes. Hence, by working together, both tanks can fill <u>one</u> tank in $\frac{17 \times 21}{17 + 21}$ minutes.

It would be pretty difficult to convince this student that the pronumeral method is easier in this case!

16. A rectangular lot is 3.5 times as long as it is wide. Its perimeter is 810 feet. How wide is the lot?

17. Jim picked a number. If he triples the number, adds 4, and then divides the sum by 8, he gets 5. What numer did he pick?

18. A trapezoid has an area of 27 square inches. If the height is 4.2 inches and one base is 5.3 inches, what is the length of the other base?

19. What principal should be invested at 3% per year to earn $90 interest in one year?

20. Three consecutive odd whole numbers add up to 63. What is the largest of these numbers?

21. The difference of one number from another is 7. If the sum of the numbers is 203, what are the numbers?

22. There are 783 pupils in Zabranchburg High School. The ratio of girls to boys is 5 to 4. How many boys are there in the school?

23. A rectangle is 43 inches long. Find its width if the length is 17 inches more than $2\frac{1}{4}$ times the width.

24. Here is an ancient riddle:

    A number and its seventh are 19.
    Find the number.

25. Delivered milk costs 25 cents a quart. This price represents a 20% increase in last year's price. What did delivered milk cost last year?

26. A tank has two inlet pipes. One pipe by itself can fill the tank in 17 minutes; the other pipe by itself can fill the tank in 21 minutes. How long will it take to fill the tank if both pipes are opened?

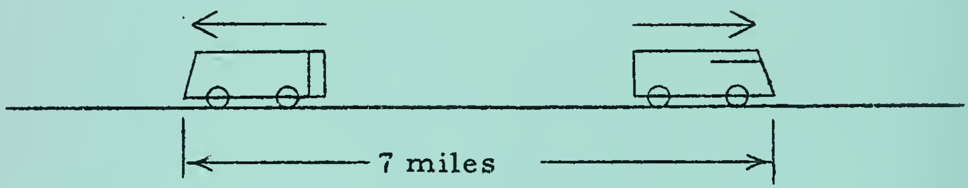
27. Bill can mow a lawn in 35 minutes and his brother can do the same job in 40 minutes. If they were to work together, how long would they take to mow the lawn?

28. Taking 50% of a certain number is the same as adding 7 to that number. What is the number?

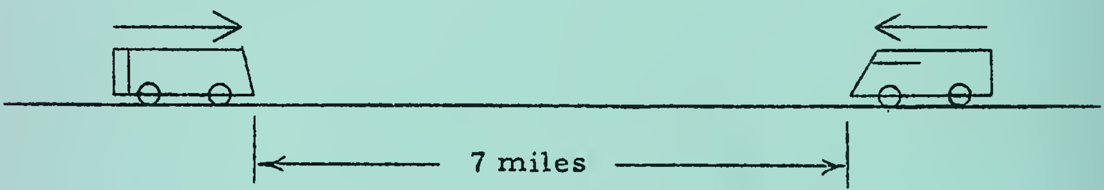
29. Andrew has twice as much money as Scott. If Andrew were to lend Scott a quarter then both boys would have the same amount of money. How much money does each boy have?

(continued on next page)

Exercise 34 is interesting. In solving this problem, you can start either by assuming:



or, by assuming:



* * *

Mrs. Catlow suggests some problems which interested her students.

1. What number when divided by $\frac{1}{3}$ of itself gives 7?

2. A circular track is 1 mile around. A kid has a hot rod that averaged 30 miles per hour for a trip half-way around the track. What would he have to average for a trip over the second half to have an over-all average of 60 miles per hour?

These problems have the same flavor as Exercise 35.

30. Two cyclists start at the same time and from the same place and travel in opposite directions. In twenty minutes they are 11 miles apart. The faster cyclist travels at an average speed which is 3 miles per hour more than the average speed of the slower cyclist. What is the average speed of each cyclist?

31. Two boys start around a 1300-foot track, running in opposite directions. If one boy runs 6 feet more per second than the other, and they meet in 24 seconds, what is the rate of the faster boy?

32. Three more than a certain number is equal to 6 less than twice the number. What is the number?

33. Edward is two years older than Charles. Eleven years ago, Edward was twice as old as Charles. How old is each boy now?

34. A freight train and a passenger train are 7 miles apart at 1:00 PM and are traveling in opposite directions. The passenger train's average speed is 35 miles per hour more than the average speed of the freight train. If they maintain their average speeds and are 45 miles apart at 1:24 PM, what is the average speed of the freight train?

35. A business man has 7 minutes to catch a train at a station which is 8 miles from his home. His taxi covers half of this distance traveling at an average speed of 30 miles per hour. What should be the average speed of the taxi during the second half of the trip to enable the man to catch the train?

36. A confectioner is making a mixture of almonds and cashews. The cashews are worth \$.90 a pound and the almonds are worth \$.75 a pound. How many pounds of each kind of nut should be used to make 30 pounds of a mixture worth \$.81 a pound?

37. A square and an equilateral triangle have equal perimeters. A side of the triangle is one inch longer than a side of the square. What is the length of a side of the square?

38. A salesman sold a number of pairs of shoes at \$8 a pair, and 5 more than that number of pairs at \$6 a pair. He received \$184 for all the shoes sold. How many pairs did he sell at each price?

(continued on next page)

Warn your students that 'quims' and 'sloogels' are UICSM "words" which do not appear in any dictionary. One of Mr. Fildes' students spent some time on research last year!

* * *

Section 3.07 deals with so-called "literal equations". You may want to tell your students that in conventional algebra courses such equations are called 'literal equations' because "they have several letters in them". However, the term is of no mathematical importance and should not be used again.

* * *

The idea of solving an equation in two pronumerals for one of the pronumerals is an extension of the student's concept of solving an equation. 'Solving an equation' should mean finding <u>numbers</u> which satisfy the equation. The phrase 'solve an equation for a pronumeral' has a larger meaning than 'solve an equation'. We agree with Professor Meserve that this difference in meaning needs mentioning when the new expression is introduced on page 2-59, and also when the student is ready to start on the exercises in Parts B and C.

39. A man has a total of \$3000 bearing interest, some at 5% and the remainder at 6%. The amount of annual interest on both investments is \$155. How much is invested at each rate?

40. A man who can row 5 miles an hour in still water rows up a stream for 3 hours and then rows back to his starting point in 2 hours. At what rate does the stream flow?

41. A man has \$3.50 in dimes and quarters. He has 17 coins in all. How many coins of each denomination does he have?

42. Divide \$155 among A, B, C, and D so that A and B together receive \$40, C receives twice as much as A, and D receives three times as much as B.

43. Mr. Smith gets a salary of \$6,000. This is \$600 more than twice the amount he earned when he left college and went to work for the first time. How much did he earn on his first job?

44. If -6 is added to half a certain number, the result is 15. What is the number?

45. Jack wants a sweater that costs \$.15 more than 3 times the amount of money he now has. If the sweater costs \$4.50, how much money does Jack have now?

46. Roger has cats and Jim has dogs. In all they have 16 cats and dogs. The number of cats that Roger has is $1\frac{2}{3}$ the number of dogs that Jim has. How many cats does Roger have?

47. Rantees has quims and Jormy has sloogels. In all they have 16 quims and sloogels. The number of quims that Rantees has is $1\frac{2}{3}$ the number of sloogels that Jormy has. How many quims does Rantees have?

3.07 Equations with different pronumerals.--Suppose you are concerned with finding the length of each of several rectangles whose perimeters and widths are known. The table at the right gives the data. In each case you could find the length by replacing the pronumerals 'P' and 'W' in the equation:

$$P = 2(L + W)$$

	Width	Perimeter
I	5''	24''
II	7''	32''
III	3''	22''
IV	10''	50''
V	9''	58''

by numerals for the perimeter and width and solving the resulting equation. Thus, in the case of rectangle I, we have:

$$\begin{aligned} 24 &= 2(L + 5) \\ 24 &= 2L + 10 \\ 24 + (-10) &= 2L + 10 + (-10) \\ 14 &= 2L \\ 14 \times \frac{1}{2} &= 2L \times \frac{1}{2} \\ 7 &= L \end{aligned}$$

The length of rectangle I is 7 inches.

Similarly, in the case of rectangle II, we have:

$$\begin{aligned} 32 &= 2(L + 7) \\ 32 &= 2L + 14 \\ 32 + (-14) &= 2L + 14 + (-14) \\ 18 &= 2L \\ 18 \times \frac{1}{2} &= 2L \times \frac{1}{2} \\ 9 &= L \end{aligned}$$

The length of rectangle II is 9 inches.

You will note that in finding the lengths of rectangles I and II we went through precisely the same procedure. The steps involved in solving the equations

'$24 = 2(L + 5)$' and '$32 = 2(L + 7)$'

followed the same pattern in each case. Indeed, if we were to find the lengths in the other cases, the steps to be followed would be the same as in the first two cases. Now, instead of going through these steps in each case, it is possible to follow the procedure just once by using pronumerals. Here is how this is accomplished:

$$\begin{aligned} P &= 2(L + W) \\ P &= 2L + 2W \\ P + (-2W) &= 2L + 2W + (-2W) \\ P - 2W &= 2L \\ (P - 2W) \times \frac{1}{2} &= 2L \times \frac{1}{2} \\ \frac{P - 2W}{2} &= L \end{aligned}$$

Now, we use this last equation which is equivalent to the original equation to find the length in each case. Note how easily we find the lengths.

Case I:

$$\begin{aligned} L &= \frac{24 - 2(5)}{2} \\ &= \frac{24 - 10}{2} \\ &= \frac{14}{2} \\ &= 7 \end{aligned}$$

The length is 7 inches.

Case II:

$$\begin{aligned} L &= \frac{32 - 2(7)}{2} \\ &= \frac{32 - 14}{2} \\ &= \frac{18}{2} \\ &= 9 \end{aligned}$$

The length is 9 inches.

Case III:

$$\begin{aligned} L &= \frac{22 - 2(3)}{2} \\ &= \frac{22 - 6}{2} \\ &= \frac{16}{2} \\ &= 8 \end{aligned}$$

The length is 8 inches.

Case IV:

$$\begin{aligned} L &= \frac{50 - 2(10)}{2} \\ &= \frac{50 - 20}{2} \\ &= \frac{30}{2} \\ &= 15 \end{aligned}$$

The length is 15 inches.

Case V:

$$\begin{aligned} L &= \frac{58 - 2(9)}{2} \\ &= \frac{58 - 18}{2} \\ &= \frac{40}{2} \\ &= 20 \end{aligned}$$

The length is 20 inches.

The process of deriving the equation:

$$L = \frac{P - 2W}{2}$$

from the equation:

$$P = 2(L + W)$$

is called <u>solving</u> '<u>P = 2(L + W)</u>' <u>for</u> '<u>L</u>'. There are many equations which contain different pronumerals. In the process of solving problems, you will frequently find it convenient to solve such equations for one of the pronumerals.

EXERCISES

<u>A</u>. In each of the following exercises you are given a table containing data about several geometric figures and asked to find the measure of a certain part of each figure. You are also given an equation which contains pronumerals related to the given data. Solve the equation for the pronumeral which is related to the part whose measure you are asked to find. Then, use the derived equation to find the measure of the part.

1.

Rectangle	Area	Width
I	28 sq. in.	4 in.
II	39 sq. in.	3 in.
III	96 sq. in.	6 in.
IV	100 sq. in.	8 in.
V	23 sq. in.	3 in.

$A = LW$

Find the length of each rectangle.

2.

$A = \frac{1}{2}hb$

Find the height of each triangle.

Triangle	Area	Base
I	72 sq. in.	12 in.
II	140 sq. in.	15 in.
III	85 sq. in.	18 in.
IV	43 sq. in.	23 in.
V	66 sq. in.	11 in.

3.

Trapezoid	Area	Height	One Base
I	66 sq. in.	6 in.	10 in.
II	64 sq. in.	8 in.	9 in.
III	3255 sq. ft.	42 ft.	73 ft.
IV	$6\frac{3}{4}$ sq. in.	$2\frac{1}{4}$ in.	$3\frac{1}{4}$ in.
V	20 sq. ft.	2.5 ft.	7.3 ft.

$A = \frac{1}{2}h(b + B)$

Find the other base of each trapezoid.

4.

$V = LWH$

Find the length of each solid.

Rectangular Solid	Volume	Height	Width
I	2730 cu. in.	13 in.	14 in.
II	3600 cu. in.	12 in.	15 in.
III	315 cu. in.	5 in.	7 in.
IV	$210\frac{15}{16}$ cu. in.	$4\frac{1}{2}$ in.	$6\frac{1}{4}$ in.
V	$122\frac{1}{2}$ cu. in.	$3\frac{1}{2}$ in.	$3\frac{1}{2}$ in.

B. Solve each equation for the indicated pronumeral.

1. '$P = 4s$' for 's'
2. '$C = 2\pi r$' for 'r'
3. '$V = Bh$' for 'h'
4. '$V = \frac{1}{3}Bh$' for 'B'
5. '$V = \pi rrh$' for 'h'
6. '$A + B + C = 180$' for 'B'
7. '$i = prt$' for 'r'
8. '$d = rt$' for 't'
9. '$A = \frac{1}{2}h(b + B)$' for 'h'
10. '$P = a + b + c$' for 'c'

In solving an equation with two or more pronumerals for one of the pronumerals, one obtains an equation which is equivalent to the given one. Consider the sample equation:

$$(1) \quad 3x - 2y = 7x + 5y + 4$$

and the derived equation:

$$(2) \quad y = \frac{-4x - 4}{7}.$$

Equations (1) and (2) are equivalent. But what do we mean by 'equivalent' when we use this word in talking about equations in <u>two</u> pronumerals? Equation (1) does not have numbers which satisfy it as does an equation in one pronumeral. Although in the next unit we shall talk in considerable detail about ordered pairs of numbers which satisfy an equation in two pronumerals, it would not be wasteful of time to explore this idea just a bit at this time.

Select a number and put a name for it in place of each 'x' in equation (1). This converts (1) into an equation in one pronumeral which can be solved. The ordered pair whose first component is given by the 'x'-replacement and whose second component is the root of the converted equation satisfies equation (1). How many such pairs are there? Do all of the ordered pairs which satisfy (1) also satisfy (2)? Conversely? The solution set of (1) is a set of ordered pairs of numbers, and is named by:

$$\{(x, y): \ 3x - 2y = 7x + 5y + 4\}.$$

The solution set of (2) is the same as that of (1), and is named by:

$$\{(x, y): \ y = \frac{-4x - 4}{7}\}.$$

Which equation, (1) or (2), is easier to use in finding the ordered pairs in the solution set?

C. Solve each of the following equations for 'y' and check your work.

Sample. $3x - 2y = 7x + 5y + 4$

Solution. We know that when we have solved this equation for 'y', we shall have an equation in which one member is 'y' and in which the other member is an expression which contains 'x' and numerals but which does not contain 'y'. So, our first step in solving for 'y' is to transform the given equation into one which contains all of the 'y'-terms in one member.

$$\begin{aligned} 3x - 2y &= 7x + 5y + 4 \\ 3x - 2y + 2y &= 7x + 5y + 4 + 2y \\ 3x &= 7x + 7y + 4 \\ 3x + (-7x) + (-4) &= 7x + 7y + 4 + (-7x) + (-4) \\ -4x - 4 &= 7y \\ (-4x - 4) \times \frac{1}{7} &= 7y \times \frac{1}{7} \\ \frac{-4x - 4}{7} &= y\ . \end{aligned}$$

As a check on our work we can replace 'x' in the original equation and in the final equation by a numeral and see if we get two equations which are equivalent.

Replace 'x' by '6' in the given equation:

$$\begin{aligned} 3(6) - 2y &= 7(6) + 5y + 4 \\ 18 - 2y &= 42 + 5y + 4 \\ 18 - 2y &= 46 + 5y \\ 18 &= 46 + 7y \\ -28 &= 7y \\ -4 &= y\ . \end{aligned}$$

A root is -4.

Replace 'x' by '6' in the final equation:

$$\begin{aligned} y &= \frac{-4(6) - 4}{7} \\ &= \frac{-24 - 4}{7} \\ &= \frac{-28}{7} \\ &= -4\ . \end{aligned}$$

In Exercise 3 of Part A, we want the student to understand that it is quite possible to have a problem which leads to a "nice" equation whose root does not give a sensible answer to the problem. Both problems lead to the same equation. In one problem the root gives a sensible answer; in the other problem, the root does not give a sensible answer. Actually, in problem (a) one is looking for a number of arithmetic and in problem (b) for a directed number. The equation has no root if the domain of the pronumeral is the set of numbers of arithmetic. In practice, one makes use of the isomorphism between the numbers of arithmetic and the non-negative directed numbers in order to treat (a) as a problem involving directed numbers. In this case, the equation does not fully formulate the problem; in addition, one needs an inequality:

$$2x + 12 = 3(x + 12) \quad \text{and} \quad x \geq 0 .$$

This equation's root is -4. We conclude that our work in solving for 'y' was probably correct.

1. $y - 7x = 15$
2. $2y + 3x = 18$
3. $5x + 2y = 6$
4. $3x - 5y = 12$
5. $4x + 2 - 6y = 9 - 3y - 5x$
6. $7x + 6y - 3 = 8y - 2x + 4$
7. $4(x - 5) + 7(y - 3) = 8x - 2y$
8. $5(x + 2) - 3(5 - y) = 9(2 - x) + 6(4 + y)$
9. $3(x + y - 2) + 7(y - x + 3) = 0$
10. $8(x - 2y) - 5(2x - y) = 9(x + 5y - 7)$

REVIEW EXERCISES

A. Follow the directions in each part below.

1. Give an example of a pair of equivalent algebraic expressions. Give an example of a pair of equivalent equations.

2. Solve the three equations:

$$x = x$$
$$x = x + x$$
$$x = x + 1$$

3. Solve the following problems:

   (a) Mary is twice as old as Bill. In 12 years she will be three times as old as Bill will be then. How old is Mary now?

   (b) According to the Weather Bureau, Nome's temperature on a certain day was twice the termperature in Anchorage. If both temperatures had increased by 12°, Nome's temperature would have been three times the temperature in Anchorage. What was Nome's temperature on that certain day?

   Discuss the solutions to these problems.

(continued on next page)

In Exercise 4 we have a problem whose data are inconsistent and a problem whose data will allow any non-negative number as an answer.

* * *

Exercise 5 provides an illustration of an error which comes up occasionally. The basis of the difficulty is the "statement":

But, we know that $x + (x + 1) + (x + 2) = 3x + 3$.

To be correct, we should say:

But, we know that <u>for every x</u>,

$$x + (x + 1) + (x + 2) = 3x + 3.$$

Since we are seeking a number x such that $x + (x + 1) + (x + 2) = 123$, we can write:

$$3x + 3 = 123.$$

* * *

In Part B, change Exercise 4 to:

For every a and every $b \neq 0$, the quotient of $2a$ by $3b$ is ________,

and Exercise 5 to:

For every $a \neq 0$, the quotient of $14a$ by $7a$ is ________ .

4. Solve both of the following problems using the pronumeral method:

   (a) In five years Mary will be the same age as she is now. How old is Mary now?

   (b) In five years Mary will be five years older than she is now. How old is Mary now?

   Discuss your solutions to these problems.

5. Consider the following problem:

   The sum of three consecutive whole numbers is 123. What are the numbers?

   We try to solve this problem as follows:

   The consecutive whole numbers are x, x + 1, and x + 2, and their sum is

   $$x + (x + 1) + (x + 2).$$

   But, we know that

   $$x + (x + 1) + (x + 2) = 3x + 3.$$

   Solve this last equation. Do you have a solution to your problem? Why?

6. A student solved an equation and found that it had the root 0. He said to himself, "Zero is nothing." So, he wrote on his paper, "The equation has no roots." What was wrong with his thinking?

B. Use the simplest expression you can to complete each of the following statements correctly.

1. For every s and t, the sum of 4s and 6t is ______.
2. For every s, the sum of 4s and 6s is ______.
3. The quotient of 75 by 10 is ______.
4. For every a and b, if $b \neq 0$, the quotient of 2a by 3b is ______.
5. For every a, if $a \neq 0$, the quotient of 14a by 7a is ______.
6. The number of toes on one foot is ______.
7. The number of toes on five feet is ______.
8. For every x, if $x \not< 0$, the number of toes on 2x + 1 feet is ______.
9. For every y, the number of feet in y yards is ______.

(continued on next page)

VII. Which sets are singletons?

1. $\{x: \ x \text{ is a whole number and } 1 < x < 3\}$

2. $\{x: \ x > 0 \ \text{ and } \ |1000 - x| = 500\}$

3. $\{x: \ |xx| = x\}$

4. $\{a: \ 3a + 2 \neq \frac{1}{2}(6a + 4)\}$

5. $\{x: \ x \neq -1 \ \text{ and } \ |7x + 4| = 3\}$

6. $\{x: \ |\frac{x}{10}| = 0\}$

7. $\{x: \ x < 2 \ \text{ and } \ \frac{|x|}{x} = 1\}$

8. $\{x: \ x < 0 \ \text{ and } \ xx - 9 = 16\}$

9. $\{x: \ 12x + 3 = 6x + 3\}$

10. $\{x: \ |5x - 2| = |2 - 5x|\}$

11. $\{y: \ (y - 3)(y - 3) = 0\}$

12. $\{k: \ \frac{k}{k} + \frac{k}{k} = 2 \ \text{ and } \ k \neq 0\}$

13. $\{t: \ t - 17.83 = 17.83 - t\}$

5. $\{x: x = 6\} = \{x: x + \frac{1}{x - 6} = 6 + \frac{1}{x - 6}\}$

6. $\{x: x + 1 = x\} \subseteq \{x: 3xx - 7x = 9xx + 3x\}$

7. $\{x: 3 + x = x + 3\} \subseteq \{x: 4x + 79(3 - x) = 8(41 - 8x)\}$

VI. Solve.

1. $-3(2x - 6) = 7x$

2. $3w = 3 + w$

3. $\frac{6}{3m + 2} = \frac{12}{4 + 6m}$

4. $\frac{3w}{6w} = \frac{1}{2}$

5. $29 + 3c = -2c + 29 - c$

6. $L + 3 = \frac{1}{3}(6 + 2L)$

7. $\frac{2s}{5} + \frac{2s}{3} = \frac{2s}{15}$

8. $\frac{t}{7} + 7 = \frac{4t}{28} - \frac{t}{7}$

9. $ax - b = bx - a$ [Solve for 'x'.]

10. $-9s + 1 = 9s - 1$

11. $3a + 5b - 2x = -(2x - 5b - 3a)$ [Solve for 'x'.]

12. $\frac{1}{x} = \frac{1}{3x}$

13. $\frac{1}{2x - 5} = \frac{-1}{5 - 2x}$

14. $aax + bbx = aabbx$ [Solve for 'x'.]

15. $2(7 - x) = \frac{-1}{6}x + 14 - \frac{10}{12}x$

16. $\frac{m}{4} + \frac{4m}{8} = \frac{5m}{12}$

17. $2y = \frac{y}{2}$

18. $-6s + 3s - 7 - 2s - 6 = -5s - 1$

19. $\frac{15s + 1}{15s - 1} = \frac{3(5s + 1)}{3(5s - 1)}$

20. $\frac{(a - b)x}{b - a} = \frac{-(a - b)}{x(a - b)}$ [Solve for 'x'.]

(continued on T. C. 64H)

8. Repeat Exercise 7, where A can do the job in x days, and B in y days.

9. If the areas of two equilateral triangles are a square inches and $\frac{a}{4}$ square inches, respectively, and the length of a side of the larger is 5s centimeters, then the length of a side of the smaller is ________ centimeters.

10. If 3 boys take x, y, and z days, respectively, to complete a job when working alone, how many days does it take them to complete this job when working together?

11. If the sum of the lengths of the two bases of a trapezoid is 3w inches, and its area is $\frac{27uw}{5}$ square inches, then the height of the trapezoid is ________ inches.

12. Jane can do a job in 7 days. Amy can work only $\frac{2}{3}$ as fast as Jane. How long would it take Amy to do this job alone? How long does it take to do the job if both girls work together?

13. If the volume of a circular cylinder is 196π cubic inches, and its height is 4aa inches, then the length of the radius of its base is ________ inches.

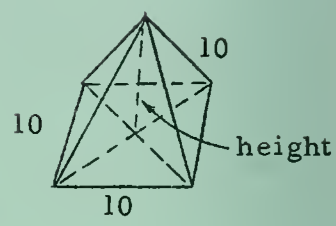
14. If m is added to $\frac{1}{p}$ of a certain number, the result is q. Find the number.

V. True or false?

1. $\{x: 3x + 7 = 9 - 2x\} = \{x: 3x + 2x = 9 - 7\}$
2. $\{x: x \neq 5 \text{ and } \frac{8}{x - 5} = 7\} = \{x: x \neq 5 \text{ and } 8 = 7(x - 5)\}$
3. $\{x: 6xx = x\} = \{x: 6x = 1\}$
4. $\{x: x = 6\} = \{x: x + 53x = 6 + 53x\}$

(continued on T. C. 64G)

9. Each of the eight edges of a pyramid with with a square base is 10 inches long. What is the height of the pyramid, what is its volume, and what is the total surface area?



IV. 1. If the base of an isosceles triangle is 6t inches and the height from this base is 4t inches, the area of the triangle is ________ square inches, and the perimeter is ________ inches.

2. If A can do $\frac{1}{6}$ of a job in 1 day, and B can do $\frac{1}{12}$ of the job in 1 day, who works faster? If A takes p days to do q of these jobs, how many days does it take B to do q of these jobs?

3. k quarts of an alcohol solution contains p% alcohol. How many quarts of alcohol should be added to make a q% solution?

4. The area of a square is 16ss square inches. If the square is cut into four congruent squares, what is the perimeter of each of these four squares?

5. If A can do $\frac{1}{4}$ of a job in 1 day, and B takes 1 day to do $\frac{1}{6}$ of this job, what part of the job can they complete in 1 day if they work together? How many days does it take to complete the entire job if they work together?

6. The edges of two cubes are 3 timps and 9 timps, respectively. If the volume of the smaller cube is 2 drubs, the volume of the larger cube is ________ drubs.

7. A can do a job in 5 days, and B can do the same job in 7 days. How long does it take them to complete the job if they work together?

(continued on T. C. 64F)

2. The difference of one number from another is 3k, and their sum is 7m. Find the numbers.

3. A rectangular lot is k times as long as it is wide. Its perimeter is p feet. Find the dimensions of the lot.

4. If A would give B m cents then they would each have n cents. How many cents does each have now?

5. If the average of two numbers is 19t, and one of the numbers is 23t, what is the other number?

6. Jim bought two cameras, one of which costs $3\frac{1}{2}$ times as much as the other. The total bill was \$39.60, including a 10% tax. What was the cost of each camera?

7. The average of three numbers is $\frac{3a}{4}$. One of them is $\frac{a}{8}$. What is the average of the other two?

8. Here is a table giving the results of a test for a mathematics class of 27 students. The column on the left lists the test scores, and the column on the right tells how many students made each of the scores. [This type of table gives what is called <u>the frequency distribution of the scores</u>.] Compute the average score (<u>the arithmetic mean</u>) and the <u>median</u> score (the middle score).

Score	Frequency
100	2
95	2
90	3
85	1
80	7
75	5
70	4
65	2
60	1

(continued on T. C. 64E)

7. 1.75m is m increased by ________% of itself.

8. 0.25m is m decreased by ________% of itself.

9. An article listed at L dollars and sold at a discount of p% is sold for ________ dollars.

10. An article listed at L dollars and sold at a "chain discount" of 10% and 30% is sold for ________ dollars.

11. If the chain discount in Exercise 10 is p% and q%, the selling price is ________ dollars.

12. If a stationer bought 20 gross of pens (list price: $2 per pen) at a 10% discount and 30 gross of pens (list price: $1.50 per pen) at a 20% discount, what %-discount did he receive for the whole transaction?

III. 1. The difference of one number from another is d, and their sum is s. Find both numbers.

<u>Solution.</u> x -- one of the numbers.

x + d -- the other number.

$$\begin{aligned} x + x + d &= s \\ 2x + d &= s \\ 2x &= s - d \\ x &= \frac{s - d}{2} \\ x + d &= \frac{s - d}{2} + d \\ &= \frac{s - d \div 2d}{2} \\ &= \frac{s + d}{2} \end{aligned}$$

The numbers are $\frac{s - d}{2}$ and $\frac{s + d}{2}$.

(continued on T. C. 64D)

11. Three years from now, Bill will be 5 years older than John. If Bill is 16 now, how old is John?

12. If the volume of a sphere is $\frac{1}{4}b$ cubic inches, and that of a cube is 2b cubic inches, then the ratio of the volume of the sphere to the volume of the cube is ________.

13. (a) If k people share equally in completing a job, what part of the job is completed by 2 of these people?

(b) If Joe can complete $\frac{1}{2}$ of a job in 3 days, what part of the job can be completed in 1 day.

(c) If k people share equally in completing a job in m days, then 1 person completes ________ of this job in m days, and 1 person completes ________ of this job in four days.

14. What is the length of the radius of a sphere whose volume is $\frac{32\pi aaa}{3}$ cubic feet?

15. If the volume of a pyramid is $\frac{hhh}{30}$ cubic inches, and its height is h inches, what is the area of its base?

II. 1. For every A, A increased by ________% of itself is 2A.

2. A increased by ________% of itself is 1.25A.

3. A decreased by 10% of itself is ________ .

4. A decreased by 101% of itself is ________ .

5. If A is decreased by 10% of itself, and the result is decreased by 10% of itself, the final result is ________ .

6. If A is increased by 50% of itself, and the result is decreased by 50% of itself, the final result is ________ .

(continued on T. C. 64C)

MISCELLANEOUS SUPPLEMENTARY EXERCISES

I. 1. A number is $\frac{1}{5}$ as big as a second number. If the first number is x, what is the second number?

2. Jim is m years old, and John is $\frac{m}{3}$ years old. Jim is _______ times as old as John.

3. Each dimension of a square of area xx square inches is reduced by $\frac{3x}{10}$ inches. What is the ratio of the area of the new square to the area of the old square?

4. If two numbers are 2x and 7x, and $x \neq 0$, the ratio of the first number to the second number is _______.

5. The difference between John's age and his father's is 30 years. John will be 12 years old next year. His father was _______ years old last year.

6. When a certain wooden ball dried up, its new diameter was $\frac{1}{12}$ of its old diameter. What is the ratio of the new volume to the old volume?

7. One number is $\frac{1}{7}$ of a second number, and the second number is 3 plus the first. What is the first number?

8. The length of the height of a parallelogram is 3p inches and the length of the base is 6p inches. What is the length of one side of a square whose area is $\frac{1}{2}$ the area of the parallelogram?

9. The sum of the ages of Joe and his brother is 25. How many years ago was the sum 19?

10. If the area of a circle is 24x square inches, and that of a square is 3x square inches, then the area of the square is _______ times the area of the circle.

(continued on T. C. 64B)

10. For every y and m, the number of feet in y yards added to the number of feet in m miles is ______.

11. ______ of 1 foot is 1 inch.

12. ______ of 1 foot is $15\frac{1}{2}$ inches.

13. For every L, ______ of 1 foot is L inches.

14. For every L, there are ______ feet in L inches.

15. 35% of 45 is ______.

16. The number which is 35% of 45 is ______.

17. The number 35% of which is 45 is ______.

18. For every x and y, if $x > y$, the perimeter of a rectangle whose dimensions are $x + y$ inches and $x - y$ inches is ______ inches.

19. The result of diminishing 107 by 3.5 is ______.

20. For every a, b, and c, the result of diminishing $a + b$ by c is ______.

21. The average of 4 and 6 is ______.

22. The average of 3.5 and 3.5 is ______.

23. For every x, the average of x and x is ______.

24. For every x and y, the average of x and y is ______.

25. For every x, y, and a, the average of 2x, 3y, and 6.2a is ______.

26. If 10 pounds of potatoes are worth 25 cents, the cost of a 60-pound bushel of potatoes is ______ cents.

27. For every c, if 10 pounds of potatoes are worth c cents, the cost of a 60-pound bushel is ______ cents.

28. For every n and c, if $n > 0$ and if n pounds of potatoes are worth c cents, the cost of a 60-pound bushel is ______ cents.

29. For every x, n, and c, if $n > 0$ and if n pounds of potatoes are worth c cents, the cost of an x-pound bushel is ______ cents.

30. For every y, the amount of interest plus principal that will result from depositing y dollars for 1 year at 6% is ______ dollars.

31. For every x and y, the amount of interest plus principal that will result from depositing y dollars for 1 year at x% is ______ dollars.

 (continued on next page)

32. For every x, the product of $\frac{1}{3}x$, $\frac{1}{2}x$, and $\frac{1}{5}x$ is ______.

33. For every x, the sum of $\frac{1}{3}x$, $\frac{1}{2}x$, and $\frac{1}{5}x$ is ______.

34. For every x, the average of $\frac{1}{3}x$, $\frac{1}{2}x$, and $\frac{1}{5}x$ is ______.

35. For every g, the number of pints in g gallons is ______.

36. For every p, the number of gallons in p pints is ______.

37. For every g and p, the number of pints in a total of p gallons and g pints is ______.

38. The absolute value of the difference between the selling prices of two articles when one article has a list price of \$50 with a 5% discount and the other article has a list price of \$55 with a 10% discount is ______.

39. For every m, n, p, and q, the absolute value of the difference between the selling prices of two articles when one article has a list price of n dollars with a p% discount and the other article has a list price of m dollars with a q% discount is ______.

40. The amount saved per year by transferring a \$3000 mortgage from a lending company which charges $5\frac{1}{4}\%$ per year to a lending company which charges $4\frac{1}{2}\%$ per year is ______ dollars.

41. For every P, r, and s, if $r > s$, the amount saved per year by transferring a P-dollar mortgage from one lending company to another and thereby changing the interest rate from r% to s% is ______ dollars.

42. If 4565 cubic feet of gas are used in a month and if gas costs \$1.06 per 1000 cubic feet, the gas bill for the month is ______ dollars.

43. For every x and p, if $0 \leq p \leq 1$, the yearly gas bill for a home using an average of x cubic feet of gas per month at a rate of $1 + p$ dollars per 1000 cubic feet is ______ dollars.

44. For every x, the result of subtracting $16\frac{2}{3}\%$ of 30 from 30% of x is ______.

C. Simplify.

1. $\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-2\right)$
2. $8 + 3 + 6(.1)$
3. $.2(6 + 3)$
4. $.2(6 - 3)$
5. $15\%(6) + 30\%(-2)$
6. $4 + x + 4 - x$
7. $\frac{1}{2}n + \frac{1}{3}n$
8. $4L \times 2L$
9. $\frac{2}{3}s \times (s + s)$
10. $9t \div \left(\frac{1}{2} - \frac{1}{3}\right)$
11. $(x - 2) - (2 - x)12$
12. $\frac{1}{6}t \times 2t$
13. $\frac{1}{4}a \times 3a \times \frac{4a}{3}$
14. $\frac{y}{7} + \frac{y}{3} + \frac{y}{21}$
15. $3a + b + 2a - 4b + a$
16. $10\%d \times 320\%d$
17. $1.6r + 2.1r - r$
18. $y + 7y - 250\%y$
19. $\frac{1}{9} + \frac{1}{2}\left(\frac{1}{9}\right) + x$
20. $\frac{1}{2}x + \frac{1}{2}y$
21. $\frac{3x}{6y}$
22. $\frac{1}{3}x + \frac{1}{2}y - \frac{1}{4}(-x)$
23. $15\%(40\%x)$
24. $4z + \frac{z}{1 + 6} - \frac{z}{6 - 5}$
25. $\left(-3\right)\left[\left(+5\right) - \left(-\frac{1}{2}\right)\right]$
26. $\frac{4t}{5} \times \left(\frac{3}{2} - \frac{7}{4}\right)$
27. $6a(2aab)$
28. $\frac{4a}{0.5} \div \frac{3}{4}$
29. $1.6w + .32w - \left(-\frac{1}{2}\right)$
30. $10\%(31\% + 22\%)q$
31. $5a(-2b)(-3c)$
32. $2(-4x)(\frac{1}{2}y)(-2z)$
33. $\frac{3aabbb}{6abb}$
34. $\frac{72xxyzzz}{28xyyzz}$

D. Expand and simplify.

1. $4(5 + 2x)$
2. $-2(x + 8)$
3. $-3(1 - x)$
4. $\left(\frac{1}{4} - \frac{1}{5}\right)\left(5x + 15x\right)$
5. $\frac{1}{2}x \times \frac{x}{5}(x - 3)$
6. $6(4 - 2x) + 2(x - 1)$
7. $\frac{1}{3}\left(\frac{1}{6}x - \frac{1}{3}y\right)$
8. $4a + 3[(-a)(1 + a)]$
9. $30\%(5L - 2)$
10. $\frac{1}{9} + \frac{1}{2}\left(x + \frac{1}{9}\right) + x$

(continued on next page)

11. $\frac{1}{8} - \frac{1}{3}\left(x - \frac{1}{8}\right) + x$

12. $2.1q(1.3q - .1)$

13. $\frac{4a + b}{2} - \frac{2b - a}{5}$

14. $-[(-x) - (-y) - .3(y - x)]$

15. $3(2x - 4y) + 7(5x + 3y)$

16. $4(3 - x) - 2(x - 5)$

17. $2(1 - 3y + x) - 5(x - y)$

18. $3(2a - b) - 4(b - 3a)$

E. Replace the pronumerals by the numerals as indicated and simplify the resulting expression.

1. $2a - 1.5b$; 'a' by '3' and 'b' by '-2'.
2. $P + Prt$; 'P' by '2000', 'r' by '.05', and 't' by '3'.
3. $\frac{4}{3}\pi rrr$; 'r' by '2.5'.
4. $\frac{1}{\frac{1}{p} + \frac{1}{q}}$; 'p' by '10' and 'q' by '15'.
5. $aa + bbb$; 'a' by '1.5' and 'b' by '-3'.
6. $\frac{(x + y)(x - y)}{(x - 2y)(x + 2y)}$; 'x' by '3' and 'y' by '-5'.

F. Solve the following equations. Check your answers.

1. $3x = 12$
2. $x + 2 = 6$
3. $x - 7 = -2$
4. $a + 4 = 3$
5. $2L = 15$
6. $4y = 19$
7. $\frac{3}{2}x = 1$
8. $4x = 4x$
9. $x - x = 1$
10. $q - 10 = 5$
11. $\frac{z}{4} = 13$
12. $\frac{9}{10}L = 90$
13. $14\%b = 3$
14. $1.7M = 2.4$
15. $3x - 3x = 0$
16. $x = 3 + x$
17. $x \div 9 = 3$
18. $8.1L = 2$
19. $125\%z = 52.1$
20. $b + 3\% = 8.1\%$
21. $\frac{L}{\frac{2}{3}} = \frac{2}{3}$
22. $a = 2a - a$
23. $\frac{2}{6}s = 15\%$
24. $8 = 3.9x$
25. $8\%t = 5$
26. $R \div (-2) = 5$

(continued on next page)

27. $-15 = 3x$

28. $\frac{4}{3}w = \frac{3}{2}$

29. $x - 1 = x - 2$

30. $-\frac{6}{7}x = \frac{36}{21}$

31. $2 + x = 1 - x$

32. $2 + x = 2 - x$

33. $5x + 5 = 6x$

34. $5x + 5 = 5x$

35. $2k + 3 = -7$

36. $r + 2r = 5r - 2r$

37. $17 - 3t = 11$

38. $5s - 11 = -9$

39. $0 - X = 0$

40. $2t + t = 5.13$

41. $y + 2 = 2 + y$

42. $y + 2 = 2 - y$

43. $25t - 7t = 6$

44. $5 + x = 15 - x$

45. $7y + 2 = 50\%$

46. $4\% x + 50 = 275$

47. $0.1x - 0.2 = 0.3x$

48. $5.2x - 7.25 = 11.75$

49. $17.4 + x = 19.7 - x$

50. $5y + .5 = 5.5$

51. $2x - 7 = 11$

52. $11x - 2 = 7$

53. $7x - 11 = 2$

54. $5x - 3 = -3$

55. $2 + 3x = 10$

56. $5 - 4x = x$

57. $2x - 7 = 7$

58. $2x + 7 = 7$

59. $-7 + 12x = 17$

60. $9x - 4 = 2$

61. $4x + 3 = 12$

62. $3 - 2x = 9$

63. $8 + x = 7 - x$

64. $9 + y = 3y - 2$

65. $-x = 7 - 2x$

66. $2 + 4x = 3x + 5$

67. $\frac{1}{2}a - 3 = 8$

68. $2 - \frac{1}{3}b = 7$

69. $2x + 7x - 3 = 15$

70. $8a - 2 + 6a = 40$

71. $5 - 6a - 2 = 8 - 3a + 5a$

72. $7 + 8x = 7x - 9x$

73. $4 + 2x = x + 4 + x$

74. $7 - \frac{1}{2}x = \frac{1}{3}x$

75. $6(1 + x) = 3(2 - x)$

76. $2(1 + x) - 4(x - 3) = 6$

77. $x + 9 = 11 + x$

78. $5\frac{1}{2}(3 + 4x) = 11$

79. $8 + x = x - \frac{1}{3}(6 - 9x)$

80. $2 - \frac{1}{4}(4 + x) = x$

(continued on next page)

81. $x + \frac{1}{2}x + \frac{1}{3}x = 2x$

82. $5 - 6x = 4x - 1$

83. $\frac{2x + 1}{4} = -8$

84. $\frac{7 - 3x}{8} = -2$

85. $\frac{x + 2}{7} = \frac{x - 3}{2}$

86. $\frac{2x - 5}{3} = \frac{4 - x}{5}$

87. $25\%(x - 7) = 30\%(2 - x)$

88. $7\frac{1}{2} + \frac{1}{3}x = 6\frac{1}{2} + 1\frac{1}{3}x$

89. $5 - 7(2 - x) = 4(3 - x)$

90. $3(x + 5) - (3x + 4) = 7(2 - 3x)$

91. $5 + 6x = \frac{1}{2}x + \frac{1}{2}(3x - 5)$

92. $7 - 2\frac{1}{2}x + 5\frac{1}{4}x = 2 + 3\frac{1}{2}(1 - 2x)$

93. $\frac{1}{2 + 3x} = 5$

94. $\frac{5 - 2x}{5 - 2x} = 3$

95. $\frac{x + 1}{x + 2} = \frac{3}{4}$

96. $5 - \frac{7 + x}{7 - x} = 2$

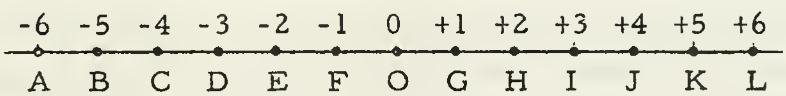
97. $8xx = 8x$
(Find 2 roots.)

98. $(2 + x)(2 + x) = 36$
(Find 2 roots.)

99. $(x - 1)(x - 2) = 0$
(Find 2 roots.)

100. $(x - 1)(x - 2)(x - 3) = 0$
(Find 3 roots.)

G. Answer the following questions using this number line.



1. What is the coordinate of the point J?
2. What is the coordinate of the point $\frac{1}{3}$ of the way from B to A?
3. What is the graph of +2?
4. What is the graph of 0?
5. Describe the location of the graph of -3.1?
6. What is the coordinate of the point 10% of the way from B to C?
7. What is the coordinate of the point 10% of the way from C to B?
8. What is the coordinate of the point halfway between H and L?
9. What is the graph of the average of -1 and -5?
10. What is the graph of the average of 10, -10, and 3?

H. Draw a number line and give the locus of each of the following expressions.

1. $x = 4$
2. $x > -3$
3. $x > 10$
4. $x + 2 = -4$
5. $3x + 2 = 6$
6. $x - 3 = 7$
7. $x - 3 > 7$
8. $x - 3 < 7$
9. $x > x$
10. $x > x - 1$
11. $x + 2 = 0$
12. $x - 7 = 0$

*13. $(x + 2)(x - 7) = 0$

*14. $(x + 1)(x - 2)(x - 4) = 0$

I. Solve each of the following equations for 'y'.

1. $2x - 7y = 18$
2. $4y + 2x = 12$
3. $x = 3y - 2$
4. $2y - 17 = x$
5. $\frac{x}{3} + \frac{y}{7} = 1$
6. $\frac{1}{2}x - \frac{2}{3}y = \frac{5}{6}$
7. $y = 3x - 7 + 2x$
8. $3(y - x) = 7x + 5 + 3y$

J. Solve the following problems. Make an estimate whenever you can.

1. Mr. Wilson bought a case of candy bars for \$12.00. There were 45 cartons in the case and each carton contained 12 bars. He sold the candy bars for 5 cents apiece. Find his margin in dollars and as a percent of the cost.

2. A man owns a house which he rents to a family for \$65.00 per month. Total maintenance (taxes, repairs, insurance, etc.) amounts to \$250.00 per year. He can sell the house for \$8500 and invest the money at 4% per year. Will it be more profitable for him to sell the house or to continue to rent it?

3. Merchandise which is sold to a company buying in quantity is often listed with two discounts. Thus, a chair with a list price of \$25.00 and discounts of 25% and 10% is first discounted 25% to \$18.75 and then this amount is discounted 10% to \$16.88. Company A lists a desk at \$85.00 and offers discounts of 35% and 5%. Company B lists the same desk at \$90.00 and offers discounts of 25% and 20%. Which company sells the desk for less money?

(continued on next page)

4. (a) Which gives the larger total discount, 10% and 15%, or 15% and 10%?

   (b) Which gives the larger total discount, 20% and 30%, or 50%?

5. A salesman sells washing machines. He receives a regular salary of \$40 a week and a commission of 5% of all weekly sales exceeding \$600. One week his total sales amounted to \$1870. How much did he earn that week?

6. The salesman in Exercise 5 received an increase in his commission rate. The next week his total sales were \$1650 and he earned \$97.75. What was his new commission rate?

7. A real estate agent sold a house for \$12,500. He charged 5% commission on the first \$10,000 and $2\frac{1}{2}$% on the remainder. What was the agent's commission? How much was received by the seller of the house?

8. What should be the selling price of the house in Exercise 7 if the same commission rate is used but the seller is to receive \$12,500?

9. A man borrowed \$45,000 which he agreed to pay back at the end of a year and 3 months with 6% annual simple interest. How much was due at the end of the term agreed upon?

10. If a man owns a \$10,000 house and insures it for 80% of its value, the insurance company will pay no more than \$8,000 (that is, 80% of \$10,000) no matter how much damage is done to the house. In this case, the \$8,000 is called the <u>insured value</u> of the house. The premium that is paid for the insurance is based on the insured value of the house. The rate for determining the amount of the premium for a year is commonly expressed as a percent of the insured value or as a certain amount to be paid per thousand dollars of insured value. For example, if the house mentioned above with an insured value of \$8,000 is insured at the rate of $\frac{1}{2}$ of 1%, then the annual premium would be \$40. If a house worth \$14,000 is insured for 75% of its value at the rate of \$4.75 per thousand dollars of insured value, what is the annual premium?

(continued on next page)

11. What is the value of a factory if it is insured for 75% of its value at a rate of $\frac{3}{4}$ of 1% and the yearly premium is $235?

12. Find the area of a triangle with a base of 10 inches and a height of 6.5 inches.

13. A triangle has an area of 170 square inches. Its base is 12 inches long. What is its height? What is its area in square feet?

14. What is the area of a circle with radius 3 inches?

15. A trapezoid has bases $\frac{9}{10}$ and $\frac{11}{10}$ inches. Its height is $\frac{6}{10}$ inches. What is the area?

16. A trapezoid has an area of 950 square inches. Its height is 25 inches and one of its bases is 25 inches. What is the length of the other base?

17. What is the area of a circle with a radius of $\frac{3}{\pi}$ inches? What is the circumference?

18. Circle I has a diameter 6 inches in length and circle II has a diameter 12 inches in length. Find the areas and the circumferences of circles I and II. What is the ratio of the diameter of circle I to the diameter of circle II? What is the ratio of the circumference of circle I to the circumference of circle II? What is the ratio of the area of circle I to the area of circle II?

19. Find the volume of a rectangular solid with edges 1 foot, 6 inches, and 3 inches.

20. A square has an area of $\frac{9}{16}$ square inches. What are the dimensions of the square?

21. If the perimeter of a square field is 360 feet, what is its area?

22. A square has the same number of inches in its perimeter as it has square inches in its area. Find its area.

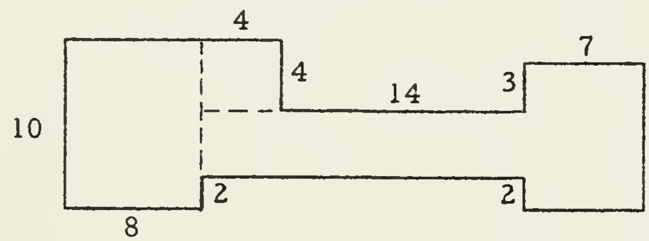
23. In mixing concrete for sidewalks and similar uses a "1-2-3" mix is often used. This means that 1 part of cement is mixed with 2 parts of sand and 3 parts of gravel (with enough water to give the proper texture). In a "1-2-3" mix if 3 buckets of cement and 9 buckets of gravel are to be used, how much sand should be

(continued on next page)

used? How much of each ingredient should be used if a total of 21 buckets (neglecting water) is to be made?

24. A man divided $3,600 among his three sons. The oldest son received 3 times as much money as the youngest. The second son received twice as much money as the youngest. How much money did each son receive?

25. Find the area of the following figure. The measurements are in inches.



26. One number is $5\frac{1}{2}$ times another number. Their sum is 39. What are the numbers?

27. Jim has a total of one hundred 2-and 3-cent stamps. Their value is $2.38. How many of each kind of stamp does he have?

28. The average of two numbers is 12. The difference between them is 4. What are the numbers?

29. One day 7% of the students enrolled at a school were absent. If 49 students were absent, what was the total number of students enrolled at the school?

30. Can you find five consecutive whole numbers having a sum 434? Explain your answer.

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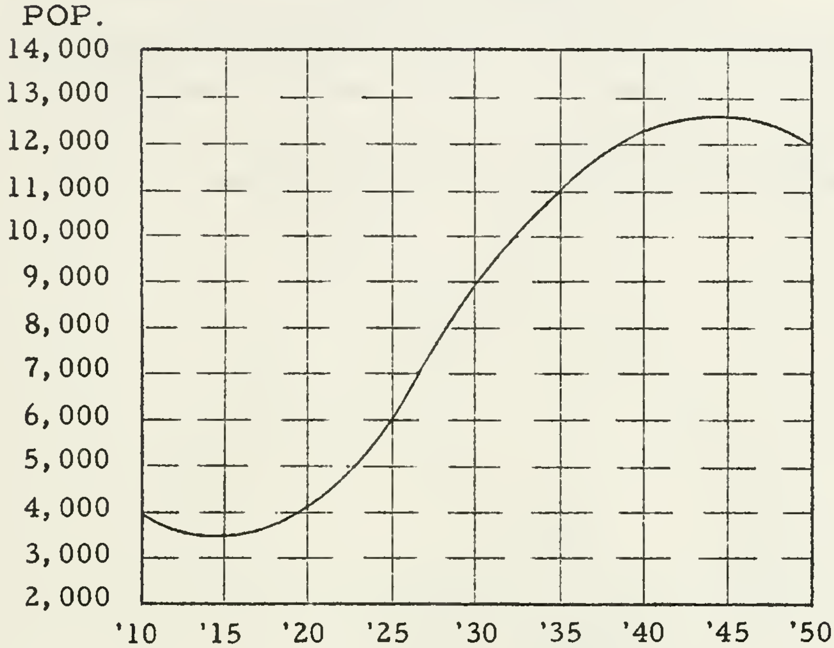
31. Use the graph to answer the following questions. (Your answers will be approximate.)

THE POPULATION OF ZABRANCHBURG

"A city of change"



(a) What was the population of Zabranchburg in 1940?

(b) What was the population of Zabranchburg in 1945?

(c) What was the population of Zabranchburg in 1918?

(d) In what year was the population of Zabranchburg 11,000?

(e) In what year was the population 5,000?

(f) When was the population 4,000?

(g) In what year was the population largest?

(h) In what year was the population smallest?

(i) Estimate the average population from 1910 to 1950.

(continued on next page)

32. Use the pie chart to answer the following questions.

(a) For what is the largest part of Zabranchburg taxes spent?

(b) For what is the smallest part of Zabranchburg taxes spent?

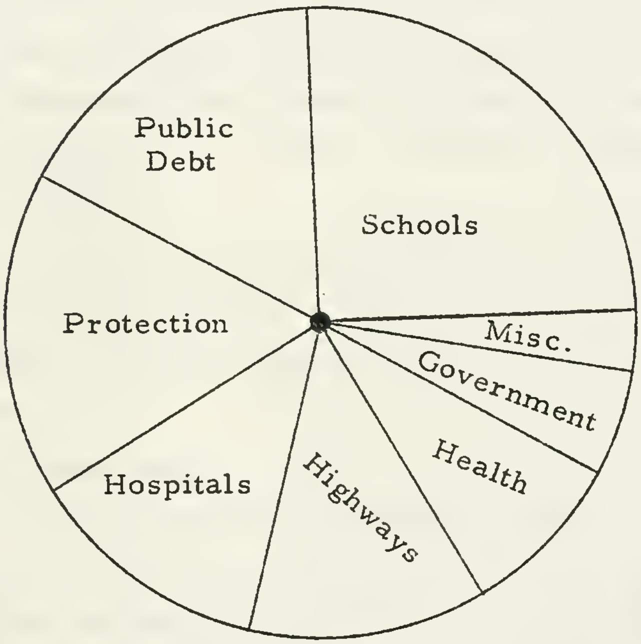
(c) For each tax dollar how many cents go to hospitals?

(d) What percent of taxes is spent on health?

(e) If total taxes is $175,000, how much goes to Misc.?

(f) What part of taxes goes to Hospitals and Health?

HOW A ZABRANCHBURG TAX DOLLAR IS SPENT



K. Estimate the root of each of the following equations by telling whether the root is larger or smaller than the number named by the right member.

Sample. $\frac{2}{3}x = 7$

Solution. Root is larger than 7.

1. $2x = 4$

2. $3x = 97$

3. $\frac{1}{2}x = \frac{3}{5}$

4. $\frac{2}{7}y = 59$

(continued on next page)

5. $x + 17 = 83$

6. $z - 92 = 27$

7. $x + 24 = 21$

8. $w - .01 = 72$

9. $.1x = 583$

10. $96\%y = 873.4$

11. $1\frac{2}{3}y = 52\frac{1}{2}$

12. $183\%k = 0.92$

L. Contributions from UICSM students.

1. The following problem was submitted by Miss Barbara Anderson who is a student in Mr. Steele's class in Blue Island:

   (Use north as the positive direction.) If a man drove 6 miles north, then 2 miles south, then 4 miles north, then 8 miles south, and then drove 5 miles one way and 3 miles in the other direction and was then a negative number of miles from his starting point, in which direction was he traveling when he drove the last 3 miles? How many miles was he from his starting point after he drove the last 3 miles?

2. Students in Miss Blair's and Miss McCoy's classes at Pekin made up the following exercises. They want you to punctuate each paragraph with single quotation marks so that the paragraph makes sense.

   (a) J. Richard Beck

   Tom and Jim were playing Scrabble. On the board was horn. Tom said, "I can use the h to make hen." Jim said, "I can use the n to make nickle." He picked up the letters a, c, k, m, and t. Tom said, "I can make made. I will use your e." He picked up the letters i, c, and e.

   (b) Cheryl Nugen

   Betty went for a walk yesterday. While she was walking, she saw some posters with President Eisenhower on them. She also saw some posters with vice-president Nixon's name on them. As she walked, she saw a sign which had Tong's Restaurant on it. She decided to go to Tong's Restaurant for lunch. However, she changed her mind when she saw Jack's Grill on a sign. She thought Jack's would be closer, so she stopped there; after her meal in Jack's Grill she went home.

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(c) Larry Augsburger

Milk is in a bottle on the table. Milk is also in the dictionary. How can milk be in both places? If we decide to transfer the milk in the bottle to a glass, that would be simple. But to get milk out of the dictionary would be very difficult. We put milk on cereal, we can also put milk on a piece of paper, and paper won't even be wet!

(d) Jim Keith

Joe and Dick were playing a game with letters. Joe said, "I can make Jerry; Jerry is just big enough." Dick then noticed that Joe could also make money. And Dick discovered that he could just fit Bill into his board. Joe said, "I like this game of yours, Dick; it's very interesting."

(e) Larry Kirgan

A man saw on the window, "One dollar for a pair of shoes". He bought the shoes for one dollar, and the clerk put one dollar on the sales slip which he gave back to the customer. Later, the man gave one dollar to a boy who was collecting for charity; the boy wrote one dollar in his record book. The man saw five dollars for a shirt, and he bought one.

(f) Lee Brecher

See eggs on that container? It has two g's in it. Inside the container are eggs. If there is eggs on the box and five eggs in the box, how many eggs could you break?

(g) Julie Anderson

Mr. Sutton, our teacher, said we were to have a home-room election. He said, "Are there any nominations for president?" The children nominated various classmates as candidates. Mr. Sutton put on the board Bill, Jane, Betty, and Tom, as they were nominated. "Who votes for Bill?" Then he put six next to Bill. "Now who wants Jane?" Five people raised their hands, so five was placed by Jane's

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name. "How many vote for Betty?" Two hands were raised, and a two was put by Betty. "Who votes for Tom?" This was the last candidate, and fifteen hands were waving in the air. Everyone knew then that Tom was the new president of his homeroom.

(h) Pat Snow

I hope we find snow at Christmas. But snow is in a different section of the dictionary. And snow would get Christmas all wet. Or if snow was found on Christmas one couldn't read Christmas very well. But Christmas wouldn't be right without snow.

(i) Bruce Sommer

Did you ever chew gum when you didn't have gum? Many people have gum but they don't chew it because they haven't any gum, really. Gum sticks to your clothing but gum doesn't. Gum has three letters. A man wrote gum on a piece of paper; then he erased it. But he found out he had gum on his paper. Don't you think this is a silly paragraph about gum that has three letters and gum that you chew?

(j) S. K. Zimmerman

The girl was making a list of things in the room. On her paper she put chair, and beside chair she put one. Next she put couch on the paper, and placed a 2 beside it. She continued until she had a complete list of chairs, couches, lamps, etc. in the room. She then counted the pieces of furniture, and found there were 23; however, she had 24 at the bottom of her list. She checked and found that she had put a 9 where she should have had an 8; she changed the 9 to 8, 10 to 9, and so on all down the list until she came to 23.